Durham University
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Epiphany 2013

Riemannian Geometry IV, Homework 1 (Week 12)

Due date for starred problems: **Tuesday, February 12**.

1.1. Lie bracket of vector fields

Let $X, Y \in \Gamma(TM)$, $X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$, $Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$. Compute the Lie bracket

$$[X, Y] = XY - YX$$

in coordinates and show that it is a vector field on M.

1.2. The equation of geodesic

Let $c(t)$ be a curve on M, and $X \in \Gamma(c^{-1}TM)$, $X(t) = \sum_{i=1}^{n} a_i(t) \frac{\partial}{\partial x_i}$. Recall that covariant derivative ∇_t of X along $c(t)$ is given by

$$\nabla_t X = \sum_{i=1}^{n} \left(a_i'(t) \frac{\partial}{\partial x_i} + a_i(t) \nabla_{c'(t)} \frac{\partial}{\partial x_i} \right)$$

Use the formula above and the definitions of connection and Christoffel symbols to show that $c(t)$ is geodesic (i.e., $\nabla_t c'(t) = 0$) if and only if for any $k = 1, \ldots, n$

$$c_k''(t) + \sum_{i,j=1}^{n} c_i'(t) c_j'(t) \Gamma_{ij}^k = 0$$

1.3. ($\star$) **Rescaling Lemma**

Let $c : [0, a] \to M$ be a geodesic, and $k > 0$. Define a curve γ by

$$\gamma : [0, a/k] \to M, \qquad \gamma(t) = c(kt)$$

Show that γ is geodesic with $\gamma'(t) = kc'(kt)$.

1.4. Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ be small enough such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $\gamma : [0, 1] \to B_\epsilon(p) \setminus \{p\}$ be any curve.

Show that there exist a curve $v : [0, 1] \to M_p$, $\|v(s)\| = 1$ for all $s \in [0, 1]$, and a non-negative function $r : [0, 1] \to \mathbb{R}_{\geq 0}$, such that

$$\gamma(s) = \exp_p(r(s)v(s))$$

1.5. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;

(b) $R(X, fY)Z = fR(X, Y)Z$;

(c) $\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle$;

(d) $R(fX, gY)hZ = fghR(X, Y)Z$.

1.6. First Bianchi Identity

Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

for X, Y, Z vector fields on M by reducing the equation to *Jacobi identity*

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

1.7. Constant sectional curvature of real hyperbolic n-space

Let $\mathbb{H}^n$ be the upper halfspace model of the real hyperbolic n-space

$$\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

Recall that the hyperbolic metric $\tilde{g}$ on $\mathbb{H}^n$ is given by

$$\tilde{g}_{ii} = \frac{1}{x_n^2}, \qquad \tilde{g}_{ij} = 0 \qquad \text{for} \quad i \neq j$$

Consider first $n = 3$.

(a) Compute the Christoffel symbols of $(\mathbb{H}^3, \tilde{g})$.

(b) Show that sectional curvatures $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ all are equal to -1 at every point of $\mathbb{H}^3$.

(c) Use (b) and the linearity of Riemann curvature tensor to show that real hyperbolic 3-space has constant sectional curvature.

Consider now the general case.

(d) Compute the Christoffel symbols of $(\mathbb{H}^n, \tilde{g})$. (The computations are very similar to (a)).

(e) Show that for every $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$ and every $i, j \in [1, \ldots, n-1]$ the submanifold

$$N = \{x \in \mathbb{H}^n \mid x_k = a_k \quad \text{for all} \quad k \neq i, j, n\}$$

with the metric induced from $\mathbb{H}^n$ is a real hyperbolic 3-space.

(f) Show that $K(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = -1$ for all pairs (i, j) at every point of $\mathbb{H}^n$.

(g) Use (f) and the linearity of Riemann curvature tensor to show that real hyperbolic n-space has constant sectional curvature.

1.8. Horosphere in hyperbolic 3-space

Consider a *horosphere*

$$M = \{x \in \mathbb{H}^3 \mid x_1^2 + x_2^2 + (x_3 - 1)^2 = 1\}$$

in real hyperbolic 3-space with metric g induced from $\mathbb{H}^3$.

(a) Parametrize M using spherical coordinates, and compute the induced metric.

(b) Compute the Christoffel symbols of (M, g).

(c) Compute the curvature tensor of (M, g). More precisely, prove that the curvature tensor is identically zero.