Durham University
Pavel Tumarkin Epiphany 2013

Riemannian Geometry IV, Homework 2 (Week 13)

Due date for starred problems: **Tuesday, February 12**.

2.1. Let (M, g) be a Riemannian manifold. The goal of this exercise is to show that M is of constant sectional curvature K_0 if and only if

$$\langle R(v_1, v_2)v_3, v_4\rangle = K_0(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

for any $p \in M$ and $v_1, v_2, v_3, v_4 \in M_p$.

Denote the expression $K_0 (\langle v_1, v_3\rangle \langle v_2, v_4\rangle - \langle v_2, v_3\rangle \langle v_1, v_4\rangle)$ by (v_1, v_2, v_3, v_4).

(a) Show that if

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in M_p$, then M is of constant sectional curvature K_0.

Now assume that M is of constant sectional curvature K_0. Our aim is to show that

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in M_p$.

(b) Show that the expression (v_1, v_2, v_3, v_4) is a tensor, i.e. it is multilinear.

(c) Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemann curvature tensor has. Namely,

- $(v_1, v_2, v_3, v_4) = -(v_2, v_1, v_3, v_4)$
- $(v_1, v_2, v_3, v_4) = -(v_1, v_2, v_4, v_3)$
- $(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$
- $(v_1, v_2, v_3, v_4) + (v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0$

(d) Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(e) Show that if no more than three distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(f) Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$

i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

(g) Use Bianchi identity to prove the initial statement.

Solution:

(a) If the equality holds, we have

$$K(v,u) = \frac{\langle R(v,u)v,u\rangle}{\|v\|^2\|u\|^2 - \langle v,u\rangle^2} = \frac{K_0(\langle v,v\rangle\langle u,u\rangle - \langle v,u\rangle\langle u,v\rangle)}{\|v\|^2\|u\|^2 - \langle v,u\rangle^2} = K_0$$

(b) This can be seen from the explicit formula for (v_1, v_2, v_3, v_4).

(c) Straighforward calculations using the definition of (v_1, v_2, v_3, v_4).

(d) By definition of sectional curvature,

$$\langle R(v,u)v,u\rangle = K_0\left(\|v\|^2\|u\|^2 - \langle v,u\rangle^2\right) = (v,u,v,u).$$

For collections of vectors ordered in other way the statement follows by (c).

(e) Using linearity and (d), we obtain

$$\begin{aligned}\langle R(v_1, v_2+v_3)v_1, v_2+v_3\rangle &= (v_1, v_2+v_3, v_1, v_2+v_3) = \\ &= (v_1,v_2,v_1,v_2) + (v_1,v_2,v_1,v_3) + (v_1,v_3,v_1,v_2) + (v_1,v_3,v_1,v_3),\end{aligned}$$

and on the other side

$$\begin{aligned}&\langle R(v_1, v_2+v_3)v_1, v_2+v_3\rangle = \\ &= \langle R(v_1,v_2)v_1,v_2\rangle + \langle R(v_1,v_2)v_1,v_3\rangle + \langle R(v_1,v_3)v_1,v_2\rangle + \langle R(v_1,v_3)v_1,v_3\rangle = \\ &= (v_1,v_2,v_1,v_2) + \langle R(v_1,v_2)v_1,v_3\rangle + \langle R(v_1,v_3)v_1,v_2\rangle + (v_1,v_3,v_1,v_3),\end{aligned}$$

which leads to

$$\langle R(v_1,v_2)v_1,v_3\rangle + \langle R(v_1,v_3)v_1,v_2\rangle = (v_1,v_2,v_1,v_3) + (v_1,v_3,v_1,v_2). \tag{1}$$

By the symmetries, we obtain

$$\langle R(v_1,v_2)v_1,v_3\rangle = \langle R(v_1,v_3)v_1,v_2\rangle,$$

and the same holds for $(\cdot,\cdot,\cdot,\cdot)$, so (1) simplifies to

$$2\langle R(v_1,v_2)v_1,v_3\rangle = 2(v_1,v_2,v_1,v_3).$$

(f) Using (e), we obtain on the one side

$$\begin{aligned}\langle R(v_1+v_4, v_2)v_3, v_1+v_4\rangle &= (v_1+v_4, v_2, v_3, v_1+v_4) = \\ &= (v_1,v_2,v_3,v_1) + (v_1,v_2,v_3,v_4) + (v_4,v_2,v_3,v_1) + (v_4,v_2,v_3,v_4),\end{aligned}$$

and on the other side

$$\begin{aligned}&\langle R(v_1+v_4, v_2)v_3, v_1+v_4\rangle = \\ &= \langle R(v_1,v_2)v_3,v_1\rangle + \langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_4,v_2)v_3,v_1\rangle + \langle R(v_4,v_2)v_3,v_4\rangle = \\ &= (v_1,v_2,v_3,v_1) + \langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_4,v_2)v_3,v_1\rangle + (v_4,v_2,v_3,v_4).\end{aligned}$$

Comparing both expressions, we conclude that

$$\langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_4,v_2)v_3,v_1\rangle = (v_1,v_2,v_3,v_4) + (v_4,v_2,v_3,v_1).$$

This implies

$$\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4) = -\langle R(v_4,v_2)v_3,v_1\rangle + (v_4,v_2,v_3,v_1).$$

Using the symmetries, we derive

$$-\langle R(v_4,v_2)v_3,v_1\rangle = -\langle R(v_3,v_1)v_4,v_2\rangle = \langle R(v_3,v_1)v_2,v_4\rangle,$$

and the same identity for $(\cdot,\cdot,\cdot,\cdot)$, so we end up with

$$\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4) = \langle R(v_3,v_1)v_2,v_4\rangle - (v_3,v_1,v_2,v_4)$$

(g) Using (f) and Bianchi identity, we conclude that

$$\begin{aligned}
3\left(\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4)\right) &= \left(\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4)\right)+\\
&+ \left(\langle R(v_3,v_1)v_2,v_4\rangle - (v_3,v_1,v_2,v_4)\right) + \left(\langle R(v_3,v_3)v_1,v_4\rangle - (v_2,v_3,v_1,v_4)\right) =\\
&= \left(\langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_3,v_1)v_2,v_4\rangle + \langle R(v_2,v_3)v_1,v_4\rangle\right)-\\
&- \left((v_1,v_2,v_3,v_4) + (v_3,v_1,v_2,v_4) + (v_2,v_3,v_1,v_4)\right) = 0-0=0,
\end{aligned}$$

which completes the proof.

2.2. $(\star)$ A Riemannian manifold (M,g) is called *Einstein manifold* if there exists $c\in\mathbb{R}$ such that

$$Ric_p(v,w) = c\langle v,w\rangle$$

for every $p\in M$, $v,w\in M_p$.

(a) Show that (M,g) is Einstein manifold if and only if there exists $c\in\mathbb{R}$ such that

$$Ric_p(v) = c$$

for every $p\in M$ and unit tangent vector $v\in M_p$.

(b) Show that if (M,g) is of constant sectional curvature then (M,g) is Einstein manifold.

Solution:

We have seen in class that $Ric_p(v,w)$ is a symmetric bilinear form on M_p, and thus $Ric_p(v)$ is a quadratic form.

(a) If M is Einstein manifold, then

$$Ric_p(v) = Ric_p(v,v) = c\langle v,v\rangle,$$

which is equal to c for any unit vector v.

Conversely, if $Ric_p(v) = c$ for any unit vector v, then, by linearity,

$$Ric_p(\lambda v) = c\lambda^2 = c\langle \lambda v,\lambda v\rangle,$$

which implies

$$Ric_p(u) = c\langle u,u\rangle$$

for arbitrary vector $u\in M_p$. Now, reconstructing symmetric bilinear form $Ric_p(v,w)$ by quadratic form $Ric_p(v) = Ric_p(v,v)$, we obtain

$$\begin{aligned}
Ric_p(v,w) &= \frac{1}{2}(Ric_p(v+w,v+w) - Ric_p(v) - Ric_p(w)) =\\
&= \frac{1}{2}(c\langle v+w,v+w\rangle - c\langle v,v\rangle - c\langle w,w\rangle) = c\langle v,w\rangle
\end{aligned}$$

(b) Let M be n-dimensional, $p\in M$, and assume $K(\Pi)=K_0$ for all 2-dimensional subspaces Π of TM. Take arbitrary unit vector $v\in M_p$, extend it to an orthonormal basis $\{v,v_2,\dots,v_n\}$. Then

$$Ric_p(v) = \sum_{i=2}^{n} K(v,v_i) = (n-1)K_0,$$

so M is Einstein manifold.

2.3. Let (M, g) be a Riemannian manifold, and ∇ be the Levi-Civita connection. Recall that a map

$$A : \Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M) \text{ or } \Gamma(TM)$$

is a *tensor* if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \Gamma(TM)$ and $f, g \in C^\infty(M)$.

(a) Let

$$T : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The *covariant derivative* of T is a map

$$\nabla T : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla T(X_1, \ldots, X_r, Y) = Y(T(X_1, \ldots, X_r)) - \textstyle\sum_{j=1}^r T(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that ∇T is a tensor.

Tensor T is called *parallel* if $\nabla T = 0$.

(b) Assume that $T_1, T_2 : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ are parallel tensors. Show that the tensor $T : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2) T_2(X_3, X_4),$$

is also parallel.

(c) Use (b) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle.$$

(d) Use (c) and Problem 2.1 to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle.$$

Solution:

(a) Linearity of ∇T on any argument can be verified by straightforward computation.

(b) We have

$$\begin{aligned}
\nabla T(X_1, X_2, X_3, X_4, Y) &= Y(T_1(X_1, X_2) T_2(X_3, X_4)) - \sum_{i=1}^4 T(X_1, \ldots, \nabla_Y X_i, \ldots, X_4) = \\
&= T_1(X_1, X_2) \underbrace{(Y(T_2(X_3, X_4)) - T_2(\nabla_Y X_3) - T_2(\nabla_Y X_4))}_{= \nabla T_2(X_3, X_4, Y) = 0} + \\
&\quad + T_2(X_3, X_4) \underbrace{(Y(T_1(X_1, X_2)) - T_1(\nabla_Y X_1) - T_1(\nabla_Y X_2))}_{= \nabla T_1(X_1, X_2, Y) = 0} = 0.
\end{aligned}$$

(c) Let $T(X, Y) = \langle X, Y\rangle$. Since ∇ is Riemannian, we have

$$\nabla T(X, Y, Z) = Z(\langle X, Y\rangle) - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0.$$

Note that $R'(X, Y, Z, W) = T(X, Z)T(Y, W) - T(X, W)T(Y, Z)$. Part (b) of the problem implies then that we have $\nabla R' = 0$.

(d) If (M, g) is a manifold with constant sectional curvature $K_0 \in \mathbb{R}$, we have by Problem 2.1

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle = K_0(\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle) = K_0 R'(X, Y, Z, W).$$

Then $\nabla R = K_0 \nabla R' = 0$ follows from (c).