Durham University
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Epiphany 2013

Riemannian Geometry IV, Homework 2 (Week 13)

Due date for starred problems: **Tuesday, February 12**.

2.1. Let (M, g) be a Riemannian manifold. The goal of this exercise is to show that M is of constant sectional curvature K_0 if and only if

$$\langle R(v_1, v_2)v_3, v_4\rangle = K_0(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

for any $p \in M$ and $v_1, v_2, v_3, v_4 \in M_p$.

Denote the expression $K_0 (\langle v_1, v_3\rangle \langle v_2, v_4\rangle - \langle v_2, v_3\rangle \langle v_1, v_4\rangle)$ by (v_1, v_2, v_3, v_4).

(a) Show that if

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in M_p$, then M is of constant sectional curvature K_0.

Now assume that M is of constant sectional curvature K_0. Our aim is to show that

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in M_p$.

(b) Show that the expression (v_1, v_2, v_3, v_4) is a tensor, i.e. it is multilinear.

(c) Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemann curvature tensor has. Namely,

- $(v_1, v_2, v_3, v_4) = -(v_2, v_1, v_3, v_4)$
- $(v_1, v_2, v_3, v_4) = -(v_1, v_2, v_4, v_3)$
- $(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$
- $(v_1, v_2, v_3, v_4) + (v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0$

(d) Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(e) Show that if no more than three distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(f) Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$

i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

(g) Use Bianchi identity to prove the initial statement.

2.2. ($\star$) A Riemannian manifold (M, g) is called *Einstein manifold* if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v, w) = c\langle v, w\rangle$$

for every $p \in M$, $v, w \in M_p$.

(a) Show that (M, g) is Einstein manifold if and only if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v) = c$$

for every $p \in M$ and unit tangent vector $v \in M_p$.

(b) Show that if (M, g) is of constant sectional curvature then (M, g) is Einstein manifold.

2.3. Let (M, g) be a Riemannian manifold, and ∇ be the Levi-Civita connection. Recall that a map

$$A : \Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M) \text{ or } \Gamma(TM)$$

is a *tensor* if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \Gamma(TM)$ and $f, g \in C^\infty(M)$.

(a) Let

$$T : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The *covariant derivative* of T is a map

$$\nabla T : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla T(X_1, \ldots, X_r, Y) = Y(T(X_1, \ldots, X_r)) - \textstyle\sum_{j=1}^r T(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that ∇T is a tensor.

Tensor T is called *parallel* if $\nabla T = 0$.

(b) Assume that $T_1, T_2 : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ are parallel tensors. Show that the tensor $T : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2)T_2(X_3, X_4),$$

is also parallel.

(c) Use (b) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle.$$

(d) Use (c) and Problem 2.1 to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle.$$