Durham University
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Riemannian Geometry IV, Homework 3 (Week 14)

Due date for starred problems: **Tuesday, February 12**.

3.1. ($\star$) Recall that Bonnet-Myers theorem implies that if (M, g) is complete, and there is $c > 0$ such that $Ric_p(v) > c$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.

Show that the assumption $c > 0$ is essential.

Hint: Consider an appropriate quadratic surface in $\mathbb{R}^3$ with induced metric.

Solution:

Let $M = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = x^2 + y^2\}$ be a paraboloid with induced metric. If we parametrize M by (x, y, x^2+y^2), an explicit computation shows that (any) curvature of M is equal to $4/(1+4x+4y)^2$, so it is positive at every point. It is obvious that M is complete and the diameter of M is infinite.

3.2. Geodesic normal coordinates

Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \dots, v_n$ be an orthonormal basis of M_p. Then a local coordinate chart of M is given by $\varphi = (x_1, \dots, x_n) : B_\epsilon(p) \to V := \{w \in \mathbb{R}^n \mid \|w\| < \epsilon\}$ via

$$\varphi^{-1}(x_1, \dots, x_n) = \exp_p(\sum_{i=1}^{n} x_i v_i).$$

The coordinate functions $x_1, \dots, x_n$ of φ are called *geodesic normal coordinates.*

(a) Let g_{ij} be the metric in terms of the above coordinate system φ. Show that at $p \in M$:

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ be an arbitrary vector, and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0,$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point $p \in M$.

Solution:

(a) We will show that

$$\frac{\partial}{\partial x_i}\Big|_p = v_i.$$

This will imply

$$g_{ij}(p) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p = \langle v_i, v_j \rangle_p = \delta_{ij}.$$

Denote by $\{e_i\}$ orthonormal basis in $V \subset \mathbb{R}^n$. Now, as $\varphi(p) = 0$, we can write

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{d}{dt}\Big|_{t=0} \varphi^{-1}(0 + te_i) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv_i) = v_i,$$

which proves (a).

(b) We have

$$c(t) = \varphi^{-1}(tw_1, \ldots, tw_n) = \exp_p(t \sum_j w_j v_j).$$

Let $v = \sum_j w_j v_j \in M_p$. Then the expression above shows that c is a geodesic with initial vector v. Let $(c_1, \ldots, c_n)|_t = \varphi(c(t))$, i.e., $c_j(t) = tw_j$, $c_j'(t) = w_j$ and $c_j''(t) = 0$. Let ∇_t denote covariant derivative along c. Since c is a geodesic, we have

$$0 = \nabla_t c' = \nabla_t \sum_j c_j' \left(\frac{\partial}{\partial x_j}(c(t)) \right) = \sum_j w_j \nabla_{c'} \frac{\partial}{\partial x_j} =$$

$$= \sum_{i,j} w_i w_j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)(c(t)) = \sum_k \left(\sum_{i,j} w_i w_j (\Gamma_{ij}^k(c(t))) \right) \frac{\partial}{\partial x_k}(c(t)).$$

Using the fact that $\frac{\partial}{\partial x_k}$ form a basis, we conclude that

$$(*) \qquad \sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0$$

for all $k \in \{1, \ldots, n\}$.

(c) Evaluating $(*)$ at $t = 0$, we obtain

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(p) = 0 \quad \text{for all } w \in \mathbb{R}^n.$$

The choice $w = e_i$ yields

$$\Gamma_{ii}^k(p) = 0,$$

and then the choice $w = e_i + e_j$ yields

$$2\Gamma_{ij}^k(p) = 0,$$

so we conclude that all Christoffel symbols vanish at p. Consequently, we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = 0.$$

3.3. Let (M, g) be a Riemannian manifold and $v_1, \ldots, v_n \in T_pM$ be an orthonormal basis. As it follows from problem 3.2, for the geodesic normal coordinates $\varphi : B_\epsilon(p) \to B_\epsilon(0) \subset \mathbb{R}^n$,

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum x_i v_i)$$

we have $\frac{\partial}{\partial x_i}|_p = v_i$ and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

Define an *orthonormal frame* $E_1, \dots, E_n : B_\epsilon(p) \to TM$ (i.e. an n-tuple of vector fields composing an orthonormal basis of M_q in every point $q \in B_\epsilon(p)$) by Gram-Schmidt orthonormalisation, i.e.,

$$\begin{aligned}
F_1(q) &:= \left.\frac{\partial}{\partial x_1}\right|_q, \qquad E_1(q) := \frac{1}{\|F_1(q)\|} F_1(q), \\
&\dots \\
F_k(q) &:= \left.\frac{\partial}{\partial x_k}\right|_q - \sum_{j=1}^{k-1} \left\langle \left.\frac{\partial}{\partial x_k}\right|_q, E_j(q) \right\rangle E_j(q), \qquad E_k(q) := \frac{1}{\|F_k(q)\|} F_k(q), \\
&\dots
\end{aligned}$$

As you might have shown in problem 3.2, $E_i(p) = v_i$ and $E_1(q), \dots, E_n(q)$ are orthonormal in M_q for all $q \in B_\epsilon(p)$. Show that

$$(\nabla_{E_i} E_j)(p) = 0$$

for all $i, j \in \{1, \dots, n\}$.

Hint: Prove first by induction over k that

$$\begin{aligned}
\left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) &= 0, &(1)\\
\nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) &= 0, &(2)\\
\left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) &= 0, &(3)
\end{aligned}$$

for all $i \in \{1, \dots, n\}$.

Solution: We prove the statements (1)–(3) by induction on k for all $i \in \{1, \dots, n\}$.

For $k = 1$ everything follows immediately from Problem 3.2(c). Assume all three equations hold for k. Then we obtain

$$\left(\nabla_{\frac{\partial}{\partial x_i}} F_{k+1}\right)(p) = \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_{k+1}}\right)(p) - \left.\frac{\partial}{\partial x_i}\right|_p \sum_{j=1}^{k} \left\langle \frac{\partial}{\partial x_{k+1}}, E_j \right\rangle E_j.$$

Using at the right hand side the product rule, the Riemannian property of the Levi-Civita connection, and the induction hypothesis $\nabla_{\frac{\partial}{\partial x_i}} E_j(p) = 0$ for $1 \le j \le k$, we conclude that the whole expression vanishes. Next, we obtain

$$\nabla_{\frac{\partial}{\partial x_i}} \langle F_{k+1}, F_{k+1} \rangle^{-1/2}(p) = -\frac{1}{\|F_{k+1}(p)\|^3} \langle \nabla_{\frac{\partial}{\partial x_i}} F_{k+1}, F_{k+1} \rangle(p)$$

(you will meet similar computation in the proof of the first variational formula of length), which implies that this expression vanishes because of (1). Finally,

$$\left(\nabla_{\frac{\partial}{\partial x_i}} E_{k+1}\right)(p) = \nabla_{\frac{\partial}{\partial x_i}} \langle F_{k+1}, F_{k+1} \rangle^{-1/2}(p) F_{k+1}(p) + \frac{1}{\|F_{k+1}(p)\|} \left(\nabla_{\frac{\partial}{\partial x_i}} F_{k+1}\right)(p),$$

which vanishes again because of (1) and (2). This completes the induction procedure.

We conclude

$$(\nabla_{E_i} E_j)(p) = 0$$

from (3), since E_i is just a linear combination of the basis vectors $\frac{\partial}{\partial x_l}$.