Durham University
Pavel Tumarkin Epiphany 2013

Riemannian Geometry IV, Homework 3 (Week 14)

Due date for starred problems: **Tuesday, February 12**.

3.1. ($\star$) Recall that Bonnet-Myers theorem implies that if (M, g) is complete, and there is $c > 0$ such that $Ric_p(v) > c$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.

Show that the assumption $c > 0$ is essential.

Hint: Consider an appropriate quadratic surface in $\mathbb{R}^3$ with induced metric.

3.2. Geodesic normal coordinates
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \ldots, v_n$ be an orthonormal basis of M_p. Then a local coordinate chart of M is given by $\varphi = (x_1, \ldots, x_n) : B_\epsilon(p) \to V := \{w \in \mathbb{R}^n \mid \|w\| < \epsilon\}$ via

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum_{i=1}^n x_i v_i).$$

The coordinate functions $x_1, \ldots, x_n$ of φ are called *geodesic normal coordinates.*

(a) Let g_{ij} be the metric in terms of the above coordinate system φ. Show that at $p \in M$:

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ be an arbitrary vector, and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0,$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point $p \in M$.

3.3. Let (M, g) be a Riemannian manifold and $v_1, \dots, v_n \in T_pM$ be an orthonormal basis. As it follows from problem 3.2, for the geodesic normal coordinates $\varphi : B_\epsilon(p) \to B_\epsilon(0) \subset \mathbb{R}^n$,

$$\varphi^{-1}(x_1, \dots, x_n) = \exp_p(\sum x_i v_i)$$

we have $\frac{\partial}{\partial x_i}|_p = v_i$ and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

Define an *orthonormal frame* $E_1, \dots, E_n : B_\epsilon(p) \to TM$ (i.e. an n-tuple of vector fields composing an orthonormal basis of M_q in every point $q \in B_\epsilon(p)$) by Gram-Schmidt orthonormalisation, i.e.,

$$\begin{aligned}
F_1(q) &:= \left.\frac{\partial}{\partial x_1}\right|_q, \qquad E_1(q) := \frac{1}{\|F_1(q)\|} F_1(q), \\
&\dots \\
F_k(q) &:= \left.\frac{\partial}{\partial x_k}\right|_q - \sum_{j=1}^{k-1} \left\langle \left.\frac{\partial}{\partial x_k}\right|_q, E_j(q) \right\rangle E_j(q), \qquad E_k(q) := \frac{1}{\|F_k(q)\|} F_k(q), \\
&\dots
\end{aligned}$$

As you might have shown in problem 3.2, $E_i(p) = v_i$ and $E_1(q), \dots, E_n(q)$ are orthonormal in M_q for all $q \in B_\epsilon(p)$. Show that

$$\left(\nabla_{E_i} E_j\right)(p) = 0$$

for all $i, j \in \{1, \dots, n\}$.

Hint: Prove first by induction over k that

$$\begin{aligned}
\left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) &= 0, \\
\nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) &= 0, \\
\left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) &= 0,
\end{aligned}$$

for all $i \in \{1, \dots, n\}$.