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Epiphany 2013

Riemannian Geometry IV, Homework 4 (Week 15)

Due date for starred problems: **Tuesday, March 12**.

4.1. ($\star$) **First Variation Formula of Length**

Let $F : (-\epsilon, \epsilon) \times [a, b] \to M$ be a variation of a differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ and X be its variational vector field. Let $l = l_F : (-\epsilon, \epsilon)$ denote the associated length functional, i.e.,

$$l(s) = \int_a^b \|\frac{\partial F}{\partial t}(s,t)\| dt.$$

(a) Show that

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \langle \nabla_s \Big|_{s=0} \frac{\partial F}{\partial t}(s,t), c'(t) \rangle \, dt$$

(b) Applying Symmetry Lemma to (a), prove the *first variational formula of length*:

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \frac{d}{dt} \langle X(t), c'(t) \rangle \, dt - \int_a^b \frac{1}{\|c'(t)\|} \langle X(t), \nabla_t c'(t) \rangle \, dt$$

Simplify the formula for the cases when

(c) c is a geodesic,

(d) F is a proper variation and c is parametrized proportionally to arc length.

(e) Show that if c is a geodesic and F is a proper variation, then $l'(0) = 0$.

(f) Let $c : [a, b] \to M$ be a differentiable curve. Show that if c is parametrized proportionally to arc length, and $l'(0) = 0$ for every proper variation of c, then c is a geodesic.

Hint: Assume c is not a geodesic. Take a smooth non-negative function $\varphi : [a, b] \to \mathbb{R}_{\geq 0}$ with $\varphi(a) = \varphi(b) = 0$ and consider a vector field along c $X(t) = \varphi(t) \nabla_t c'(t)$. Using the fact that $X(t)$ is a variational vector field for some proper variation F of c, find appropriate φ such that $l'_F(0) \neq 0$.

Solution:

(a)

$$l'(s) = \frac{d}{ds} \int_a^b \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \rangle^{1/2} \, dt = \int_a^b \frac{d}{ds} \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \rangle^{1/2} \, dt =$$

$$= \int_a^b \frac{1}{2} \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \rangle^{1/2} \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \rangle \, dt =$$

$$= \int_a^b \frac{1}{2\|\frac{\partial F}{\partial t}(s,t)\|} \left(\langle \nabla_s \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \rangle + \langle \frac{\partial F}{\partial t}(s,t), \nabla_s \frac{\partial F}{\partial t}(s,t) \rangle \right) dt =$$

$$= \int_a^b \frac{1}{\|\frac{\partial F}{\partial t}(s,t)\|} \langle \nabla_s \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \rangle \, dt$$

Since $\frac{\partial F}{\partial t}(0,t) = c'(t)$, we obtain

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \langle \nabla_s \Big|_{s=0} \frac{\partial F}{\partial t}(s,t), c'(t) \rangle \, dt$$

(b) Applying Symmetry Lemma to (a), we get

$$\begin{aligned} l'(0) &= \int_a^b \frac{1}{\|c'(t)\|} \langle \nabla_s \Big|_{s=0} \frac{\partial F}{\partial t}(s,t), c'(t) \rangle \, dt = \int_a^b \frac{1}{\|c'(t)\|} \langle \nabla_t \frac{\partial F}{\partial s}(0,t), c'(t) \rangle \, dt = \\ &= \int_a^b \frac{1}{\|c'(t)\|} \left(\frac{d}{dt} \langle \frac{\partial F}{\partial s}(0,t), c'(t) \rangle - \langle \frac{\partial F}{\partial s}(0,t), \nabla_t c'(t) \rangle \right) dt = \\ &= \int_a^b \frac{1}{\|c'(t)\|} \frac{d}{dt} \langle X(t), c'(t) \rangle \, dt - \int_a^b \frac{1}{\|c'(t)\|} \langle X(t), \nabla_t c'(t) \rangle \, dt, \end{aligned}$$

where $X(t) = \frac{\partial F}{\partial s}(0,t)$.

(c) If $c(t)$ is geodesic, then $\|c'(t)\|$ is constant, and $\nabla_t c'(t) = 0$. Thus,

$$l'(0) = \frac{1}{\|c'(t)\|} \int_a^b \frac{d}{dt} \langle X(t), c'(t) \rangle \, dt = \frac{1}{\|c'(t)\|} \big(\langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle \big)$$

(d) If F is a proper variation, and $c(t)$ is parametrized by arc lenth, then $\|c'(t)\|$ is constant, and $X(a) = X(b) = 0$. Therefore,

$$\begin{aligned} l'(0) &= \frac{1}{\|c'(t)\|} \int_a^b \frac{d}{dt} \langle X(t), c'(t) \rangle \, dt - \frac{1}{\|c'(t)\|} \int_a^b \langle X(t), \nabla_t c'(t) \rangle \, dt = \\ &= \frac{1}{\|c'(t)\|} \big(\langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle \big) - \frac{1}{\|c'(t)\|} \int_a^b \langle X(t), \nabla_t c'(t) \rangle \, dt = \\ &= -\frac{1}{\|c'(t)\|} \int_a^b \langle X(t), \nabla_t c'(t) \rangle \, dt \end{aligned}$$

(e) If F is a proper variation and $c(t)$ is geodesic, then we have

$$l'(0) = \frac{1}{\|c'(t)\|} \big(\langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle \big) = 0$$

(f) Assume $c(t)$ is not a geodesic, but it is parametrized proportionally to arc length. Then at some point $t_0 \in (a,b)$ we have $\nabla_t c'(t) \neq 0$. Take a smooth non-negative function $\varphi : [a,b] \to \mathbb{R}_{\geq 0}$ with $\varphi(a) = \varphi(b) = 0$, $\varphi(t_0) \neq 0$, and consider a vector field $X(t)$ along $c(t)$ defined by $X(t) = \varphi(t) \nabla_t c'(t)$. Let F be a proper variation of c with variational vector field $X(t)$, denote by $l_F(s)$ the length functional of F. We want to show that $l'_F(0) \neq 0$. Indeed, by (d) we have

$$\begin{aligned} l'_F(0) &= -\frac{1}{\|c'(t)\|} \int_a^b \langle X(t), \nabla_t c'(t) \rangle \, dt = -\frac{1}{\|c'(t)\|} \int_a^b \langle \varphi(t) \nabla_t c'(t), \nabla_t c'(t) \rangle \, dt = \\ &= -\frac{1}{\|c'(t)\|} \int_a^b \varphi(t) \langle \nabla_t c'(t), \nabla_t c'(t) \rangle \, dt = -\frac{1}{\|c'(t)\|} \int_a^b \varphi(t) \|\nabla_t c'(t)\|^2 \, dt < 0 \end{aligned}$$

4.2. Let M , N be smooth manifolds of dimension m and n respectively, and let $f : M \to N$ be a smooth map. Take $p \in M$.

(a) Show that the differential $df_p : M_p \to N_{f(p)}$ is a linear map.

Now let $M = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1,\ |z| < 1\}$ be a cylinder, and $N = S^2 \subset \mathbb{R}^3$ be the unit sphere, both with the metric induced by $\mathbb{R}^3$. Define $f : M \to N$ by

$$f(x, y, z) = (x\sqrt{1 - z^2}, y\sqrt{1 - z^2}, z)$$

Parametrize M and N by cylindrical coordinates (φ, z) and spherical coordinates (φ, ϑ) respectively.

(b) Write down the equation of a geodesic on M through (φ_0, z_0) in the direction $a\frac{\partial}{\partial\varphi} + b\frac{\partial}{\partial z}$. (*Hint*: do not compute anything!)

(c) Compute the matrix of the differential of f in the bases $(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z})$ and $(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})$.

Solution:

(a) This immediately follows from the definition of the differential (or can be easily computed in coordinates).

We parametrize the cylinder M by $(x = \cos\varphi, y = \sin\varphi, z)$, and the sphere N by $(x = \cos\varphi\sin\vartheta, y = \sin\varphi\sin\vartheta, z = \cos\vartheta)$.

(b) It easy to see that the metric on M is Euclidean since the cylinder is obtained from a plane by a "bending" which does not change its geometry (this can also be easily verified by the calculation of the metric). Thus, the geodesics are the images of geodesics on the plane, i.e. the curves whose angle with vertical lines on M is constant. In other words, z should be a linear function of φ. Therefore, such a geodesic is given by

$$\gamma(t) = (at + \varphi_0, bt + z_0)$$

(c) Since $z = \cos\vartheta$, the function f maps (φ, z) to $(\varphi, \vartheta = \arccos z)$. Thus, the only partial derivative we need to compute is $\frac{\partial\vartheta}{\partial z} = -\frac{1}{\sqrt{1-z^2}} = -\frac{1}{\sin\vartheta}$, and the matrix of the differential $df_{(\varphi,z)}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\sin\vartheta} \end{pmatrix}$$

This can be also seen by the definition.

Choose a point $(\varphi, z) \in M$. Let

$$c_z(t) = (\varphi, z + t) = (\cos\varphi, \sin\varphi, z + t), \qquad c_\varphi(t) = (\varphi + t, z) = (\cos(\varphi + t), \sin(\varphi + t), z)$$

Then

$$\frac{\partial}{\partial z} = c_z'(0), \qquad \frac{\partial}{\partial\varphi} = c_\varphi'(0)$$

Computing $\frac{d}{dt}\Big|_{t=0} f(c_\varphi(t))$ and $\frac{d}{dt}\Big|_{t=0} f(c_z(t))$, we obtain

$$df_{(\varphi,z)}\frac{\partial}{\partial\varphi} = \frac{d}{dt}\Big|_{t=0} f(c_\varphi(t)) = \frac{d}{dt}(\cos(\varphi + t)\sin\vartheta, \sin(\varphi + t)\sin\vartheta, \cos\vartheta) = \frac{\partial}{\partial\varphi},$$

$$\begin{aligned} df_{(\varphi,z)}\frac{\partial}{\partial z} &= \frac{d}{dt}\Big|_{t=0} f(c_z(t)) = \frac{d}{dt}\Big|_{t=0}(\cos\varphi\sqrt{1 - (z + t)^2}, \sin\varphi\sqrt{1 - (z + t)^2}, z + t) = \\ &= (\cos\varphi\frac{-(z + t)}{\sqrt{1 - (z + t)^2}}, \sin\varphi\frac{-(z + t)}{\sqrt{1 - (z + t)^2}}, 1)\Big|_{t=0} = (\cos\varphi\frac{-z}{\sqrt{1 - z^2}}, \sin\varphi\frac{-z}{\sqrt{1 - z^2}}, 1) = \\ &= (\cos\varphi\frac{-\cos\vartheta}{\sin\vartheta}, \sin\varphi\frac{-\cos\vartheta}{\sin\vartheta}, 1) = -\frac{1}{\sin\vartheta}(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta) = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta} \end{aligned}$$