Durham University
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Epiphany 2013

Riemannian Geometry IV, Homework 4 (Week 15)

Due date for starred problems: **Tuesday, March 12**.

4.1. ($\star$) **First Variation Formula of Length**
Let $F : (-\epsilon, \epsilon) \times [a, b] \to M$ be a variation of a differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ and X be its variational vector field. Let $l = l_F : (-\epsilon, \epsilon)$ denote the associated length functional, i.e.,

$$l(s) = \int_a^b \|\frac{\partial F}{\partial t}(s,t)\| dt.$$

(a) Show that

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \langle \nabla_s\Big|_{s=0} \frac{\partial F}{\partial t}(s,t), c'(t)\rangle \, dt$$

(b) Applying Symmetry Lemma to (a), prove the *first variational formula of length*:

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \frac{d}{dt} \langle X(t), c'(t)\rangle \, dt - \int_a^b \frac{1}{\|c'(t)\|} \langle X(t), \nabla_t c'(t)\rangle \, dt$$

Simplify the formula for the cases when

(c) c is a geodesic,

(d) F is a proper variation and c is parametrized proportionally to arc length.

(e) Show that if c is a geodesic and F is a proper variation, then $l'(0) = 0$.

(f) Let $c : [a, b] \to M$ be a differentiable curve. Show that if c is parametrized proportionally to arc length, and $l'(0) = 0$ for every proper variation of c, then c is a geodesic.

Hint: Assume c is not a geodesic. Take a smooth non-negative function $\varphi : [a, b] \to \mathbb{R}_{\geq 0}$ with $\varphi(a) = \varphi(b) = 0$ and consider a vector field along c $X(t) = \varphi(t) \frac{D}{dt} c'(t)$. Using the fact that $X(t)$ is a variational vector field for some proper variation F of c, find appropriate φ such that $l_F'(0) \neq 0$.

4.2. Let M , N be smooth manifolds of dimension m and n respectively, and let $f : M \to N$ be a smooth map. Take $p \in M$.

(a) Show that the differential $df_p : M_p \to N_{f(p)}$ is a linear map.

Now let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \ |z| < 1\}$ be a cylinder, and $N = S^2 \subset \mathbb{R}^3$ be the unit sphere, both with the metric induces by $\mathbb{R}^3$. Define $f : M \to N$ by

$$f(x, y, z) = (x\sqrt{1 - z^2}, y\sqrt{1 - z^2}, z)$$

Parametrize M and N by cylindrical coordinates (φ, z) and spherical coordinates (φ, ϑ) respectively.

(b) Write down the equation of a geodesic on M through (φ_0, z_0) in the direction $a\frac{\partial}{\partial \varphi} + b\frac{\partial}{\partial z}$. (*Hint*: do not compute anything!)

(c) Compute the matrix of the differential of f in the bases $(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z})$ and $(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \vartheta})$.