Durham University
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Riemannian Geometry IV, Homework 5 (Week 16)

Due date for starred problems: **Tuesday, March 12**.

5.1. Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be the unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2, \pi/2] \to TS^2$ along c by

$$X(t) = (0, \cos t, 0).$$

Let ∇_t denote covariant derivative on S^2 along c.

(a) Calculate $\nabla_t X(t)$ and $\nabla_t^2 X(t)$.

(b) Show that X satisfies the Jacobi equation.

Solution:

One way to solve the problem is by direct computation: compute Christoffel symbols, and then compute first and second covariant derivative of $X(t)$ and verify the Jacobi equation for $X(t)$.

Another way makes use of the following observation (which is of interest itself). Let $M \subset \mathbb{R}^3$ have metric induced from $\mathbb{R}^3$, $p \in M$, and denote by π_p the orthogonal projection from $\mathbb{R}^3_p$ to M_p. Let X be a vector field on M, $\tilde{X}$ its extension in $\mathbb{R}^3$, denote by $\tilde{\nabla}$ the Levi-Civita connection on $\mathbb{R}^3$: for $u \in \mathbb{R}^3_p$

$$\tilde{\nabla}_u \tilde{X} = \lim_{s \to 0} \frac{\tilde{X}(p + su) - \tilde{X}(p)}{s}$$

Then for any $v \in M_p$ Levi-Civita connection ∇ on M can be defined by

$$(*) \qquad \nabla_v X = \pi_p(\tilde{\nabla}_v \tilde{X})$$

This can be verified in the following way: the expression $\nabla_v X$ defined by $(*)$ satisfies all the properties of Levi-Civita connection, thus it coincides with Levi-Civita connection on M by the uniqueness property.

Now let us apply the observation above to our problem.

We obtain

$$\nabla_t X(t) = \left(\frac{d}{dt}(0, \cos t, 0) \right)^{\perp} = (0, -\sin t, 0)^{\perp},$$

where the orthogonal projection $v^{\perp}$ of v onto S^2_p is taken at the point $c(t) = (\cos t, 0, \sin t)$. Since $(0, -\sin t, 0) \perp c(t)$, we get

$$\nabla_t X(t) = (0, -\sin t, 0).$$

Similarly, we conclude that

$$\nabla_t^2 X(t) = (0, -\cos t, 0) = -X(t).$$

Now, since S^2 has constant curvature 1, we make use of the result of Problem 2.1 to obtain (cf. Problem 5.2(a))

$$R(c'(t), X(t))c'(t) = \langle c'(t), c'(t) \rangle X(t) + \langle X(t), c'(t) \rangle c'(t) = 1 \cdot X(t) + 0 \cdot c'(t) = X(t)$$

Bringing everything together, we conclude that

$$\nabla_t^2 X(t) + R(c'(t), X(t))c'(t) = -X(t) + X(t) = 0,$$

i.e., X satisfies the Jacobi equation.

5.2. ($\star$) **Jacobi fields on manifold of constant curvature.**

Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic satisfying $\|c'\| = 1$. Let $J : [0,1] \to TM$ be an orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t)\rangle = 0$ for every $t \in [0,1]$).

(a) Show that $R(c', J)c' = KJ$. *Hint:* You may use result of Problem 2.1.

(b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \nabla_t J(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} Z_2(t) & \text{if } K < 0. \end{cases}$$

Hint: Show that these fields satisfy Jacobi equation, there value and covariant derivative at $t = 0$ is the same as for $J(t)$, and then use uniqueness (Corollary 3.7).

Solution:

(a) We conclude from Problem 2.1 that

$$R(v_1, v_2)v_3 = K(\langle v_1, v_3\rangle v_2 - \langle v_2, v_3\rangle v_1).$$

This implies

$$R(c', J)c' = K(\langle c', c'\rangle J - \langle J, c'\rangle c').$$

Since $\|c'\|^2 = 1$ and $J \perp c'$, we obtain

$$R(c', J)c' = KJ.$$

(b) We only consider the case $K > 0$, all other cases are similar. The vector field

$$J_1(t) = \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t)$$

satisfies $J_1(0) = Z_1(0) = J(0)$ and

$$\nabla_t J_1(t) = -\sqrt{K}\sin(t\sqrt{K})Z_1(t) + \cos(t\sqrt{K})Z_2(t),$$

which implies $\nabla_t J_1(0) = Z_2(0) = \nabla_t J(0)$. Obviously, we have

$$\nabla_t^2 J_1(t) = -K\cos(t\sqrt{K})Z_1(t) - \sqrt{K}\sin(t\sqrt{K})Z_2(t) = -KJ_1(t),$$

and therefore we obtain

$$\nabla_t^2 J_1(t) + KJ_1(t) = 0$$

Now we want to apply (a) to say that the equation above is the Jacobi equation. For this, we need to show that the field J_1 is orthogonal. It suffices to demonstrate that both Z_1 and Z_2 are orthogonal.

Indeed, since Z_i are parallel, $\langle Z_i, c'\rangle$ is constant, so we compute the values of $\langle Z_i(t), c'(t)\rangle$ at $t = 0$. We have

$$\langle Z_1(0), c'(0)\rangle = \langle J(0), c'(0)\rangle = 0$$

since $J(t)$ is orthogonal, and

$$\langle Z_2(0), c'(0)\rangle = \langle \nabla_t J(0), c'(0)\rangle = 0$$

since $J(t)$ is orthogonal, which completes the proof.