Durham University
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Epiphany 2013

Riemannian Geometry IV, Homework 5 (Week 16)

Due date for starred problems: **Tuesday, March 12**.

5.1. Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be the unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2, \pi/2] \to TS^2$ along c by

$$X(t) = (0, \cos t, 0).$$

Let ∇_t denote covariant derivative on S^2 along c.

(a) Calculate $\nabla_t X(t)$ and $\nabla_t^2 X(t)$.

(b) Show that X satisfies the Jacobi equation.

5.2. ($\star$) **Jacobi fields on manifold of constant curvature.**
Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic satisfying $\|c'\| = 1$. Let $J : [0,1] \to TM$ be an orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t)\rangle = 0$ for every $t \in [0,1]$).

(a) Show that $R(c', J)c' = KJ$. *Hint:* You may use result of Problem 2.1.

(b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \nabla_t J(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} Z_2(t) & \text{if } K < 0. \end{cases}$$

Hint: Show that these fields satisfy Jacobi equation, there value and covariant derivative at $t = 0$ is the same as for $J(t)$, and then use uniqueness (Corollary 3.7).