Durham University
Pavel Tumarkin Epiphany 2013

Riemannian Geometry IV, Homework 6 (Week 17)

Due date for starred problems: **Tuesday, March 12**.

6.1. ($\star$) Let M be a Riemannian manifold of non-positive sectional curvature.

(a) Let $c : [a, b] \to M$ be a differentiable curve and J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \geq 0$, i.e., f is a convex function.

(b) Derive from (a) that M does not admit conjugate points.

Solution:

(a) We have

$$f'(t) = \frac{d}{dt}\Big|_{t=0} \langle J(t), J(t)\rangle = 2\langle \nabla_t J(t), J(t)\rangle$$

and

$$f''(t) = 2\left(\langle \nabla_t^2 J(t), J(t)\rangle + \left\|\nabla_t J(t)\right\|^2\right).$$

Using Jacobi's equation, we conclude

$$f''(t) = 2\left(-\langle R(c'(t), J(t))c'(t), J(t)\rangle + \left\|\nabla_t J(t)\right\|^2\right).$$

We have $\langle R(c'(t), J(t))c'(t), J(t)\rangle = 0$ if $J(t), c'(t)$ are linear dependent and, otherwise, for $\Pi = \text{span}(J(t), c'(t)) \subset M_{c(t)}$,

$$\langle R(c'(t), J(t))c'(t), J(t)\rangle = K(\Pi)\left(\|J(t)\|^2\|c'(t)\|^2 - (\langle J(t), c'(t)\rangle)^2\right) \leq 0,$$

since sectional curvature is non-positive (and using Cauchy-Schwarz inequality). This shows that $f''(t)$, as a sum of two non-negative terms, is greater than or equal to zero.

(b) If there were a conjugate point $q = c(t_2)$ to a point $p = c(t_1)$ along the geodesic c, then we would have a non-vanishing Jacobi field J along c with $J(t_1) = 0$ and $J(t_2) = 0$. This would imply that the convex, non-negative function $f(t) = \|J(t)\|^2$ would have zeros at $t = t_1$ and $= t_2$. This would force f to vanish identically on the interval $[t_1, t_2]$, which would imply that J vanishes as well, which leads to a contradiction.

6.2. Jacobi fields and conjugate points on locally symmetric spaces

A Riemannian manifold (M, g) is called a *locally symmetric space* if $\nabla R = 0$ (see Problem 2.3). Let (M, g) be an n-dimensional locally symmetric space and $c : [0, \infty) \to M$ be a geodesic with $p = c(0)$ and $v = c'(0) \in M_p$. Prove the following facts:

(a) Let X, Y, Z be parallel vector fields along c. Show that $R(X, Y)Z$ is also parallel.

(b) Let $K_v \in \text{Hom}\,(M_p, M_p)$ be the curvature operator, defined by

$$K_v(w) = R(v, w)v.$$

Show that K_v is self-adjoint, i.e.,

$$\langle K_v(w_1), w_2\rangle = \langle w_1, K_v(w_2)\rangle$$

for every pair of vectors $w_1, w_2 \in M_p$.

(c) Choose an orthonormal basis $w_1, \ldots, w_n \in M_p$ that diagonalizes K_v, i.e.,

$$K_v(w_i) = \lambda_i w_i.$$

(such a basis exists since K_v is self-adjoint). Let $W_1, \ldots, W_n$ be the parallel vector fields along c with $W_i(0) = w_i$. Show that, for all $t \in [0, \infty)$,

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).$$

(d) Let $J(t) = \sum_i J_i(t) W_i(t)$ be a Jacobi field along c. Show that Jacobi's equation translates into

$$J_i''(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \ldots, n.$$

(e) Show that the conjugate points of p along c are given by $c(\pi k/\sqrt{\lambda_i})$, where k is any positive integer and λ_i is a positive eigenvalue of K_v.

Solution:

(a) We know that $\nabla R = 0$. Let ∇_t denote covariant derivative along c. Then we have, for parallel vector fields X, Y, Z along c that

$$\begin{aligned} 0 = \nabla R(X, Y, Z, c')(t) &= \nabla_t R(X(t), Y(t))Z(t) - \\ &- R(\underbrace{\nabla_t X(t)}_{=0}, Y(t))Z(t) - R(X(t), \underbrace{\nabla_t Y(t)}_{=0})Z(t) - R(X(t), Y(t)) \underbrace{\nabla_t Z(t)}_{=0} = \\ &= \nabla_t R(X(t), Y(t))Z(t). \end{aligned}$$

This shows that $R(X, Y)Z$ is parallel.

(b) The symmetries of R yield

$$\langle K_v(w_1), w_2 \rangle = \langle R(v, w_1)v, w_2 \rangle = \langle R(v, w_2)v, w_1 \rangle = \langle K_v(w_2), w_1 \rangle = \langle w_1, K_v(w_2) \rangle.$$

(c) Since K_v is self-adjoint, we can find an orthonormal basis $w_1, \ldots, w_n \in M_p$ with $K_v(w_i) = \lambda_i w_i$. We know, by (a), that $K_{c'(t)}(W_i(t)) = R(c'(t), W_i(t))W_i(t)$ is parallel and, since $K_{c'(0)}(W_i(0)) = K_v(w_i) = \lambda_i w_i$, we must have

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t),$$

since parallel vector fields V along c are uniquely determined by their initial values $V(0) \in M_p$.

(d) Let J be a Jacobi field along c. Then J satisfies the Jacobi equation

$$\nabla_t^2 J + R(c', J)c' = 0.$$

Since $W_1, \ldots, W_n$ form a parallel basis along c, we obtain, by taking inner product with W_i:

$$\begin{aligned} 0 = \langle \nabla_t^2 J, W_i \rangle + \langle R(c', J)c', W_i \rangle &= \\ &= \frac{d^2}{dt^2} \sum_j J_j \langle W_j, W_i \rangle + \sum_j J_j \langle R(c', W_j)c', W_i \rangle = \\ &= J_i'' + \sum_j J_j \lambda_j \langle W_j, W_i \rangle = J_i'' + \lambda_i J_i. \end{aligned}$$

(e) The unique solution of $J_i''(t) + \lambda_i J_i(t) = 0$, $J_i(0) = 0$ (up to scalar multiples) is given by

$$J_i(t) = \begin{cases} \sin(t\sqrt{\lambda_i}) & \text{if } \lambda_i > 0, \\ t & \text{if } \lambda_i = 0, \\ \sinh(t\sqrt{-\lambda_i}) & \text{if } \lambda_i < 0. \end{cases}$$

So J_i has zeros for positive t only if $\lambda_i > 0$, and these are precisely at $t = \pi k/\sqrt{\lambda_i}$. The corresponding Jacobi fields with $J(0) = 0$ and $\nabla_t J(0) = w_i$ produce the conjugate points $c(\pi k/\sqrt{\lambda_i})$.