Durham University
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Epiphany 2013

Riemannian Geometry IV, Homework 6 (Week 17)

Due date for starred problems: **Tuesday, March 12**.

6.1. ($\star$) Let M be a Riemannian manifold of non-positive sectional curvature.

(a) Let $c : [a, b] \to M$ be a differentiable curve and J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \geq 0$, i.e., f is a convex function.

(b) Derive from (a) that M does not admit conjugate points.

6.2. Jacobi fields and conjugate points on locally symmetric spaces
A Riemannian manifold (M, g) is called a *locally symmetric space* if $\nabla R = 0$ (see Problem 2.3). Let (M, g) be an n-dimensional locally symmetric space and $c : [0, \infty) \to M$ be a geodesic with $p = c(0)$ and $v = c'(0) \in M_p$. Prove the following facts:

(a) Let X, Y, Z be parallel vector fields along c. Show that $R(X, Y)Z$ is also parallel.

(b) Let $K_v \in \mathrm{Hom}\,(M_p, M_p)$ be the curvature operator, defined by

$$K_v(w) = R(v, w)v.$$

Show that K_v is self-adjoint, i.e.,

$$\langle K_v(w_1), w_2 \rangle = \langle w_1, K_v(w_2) \rangle$$

for every pair of vectors $w_1, w_2 \in M_p$.

(c) Choose an orthonormal basis $w_1, \ldots, w_n \in M_p$ that diagonalizes K_v, i.e.,

$$K_v(w_i) = \lambda_i w_i.$$

(such a basis exists since K_v is self-adjoint). Let $W_1, \ldots, W_n$ be the parallel vector fields along c with $W_i(0) = w_i$. Show that, for all $t \in [0, \infty)$,

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).$$

(d) Let $J(t) = \sum_i J_i(t) W_i(t)$ be a Jacobi field along c. Show that Jacobi's equation translates into

$$J_i''(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \ldots, n.$$

(e) Show that the conjugate points of p along c are given by $c(\pi k/\sqrt{\lambda_i})$, where k is any positive integer and λ_i is a positive eigenvalue of K_v.