Durham University
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Epiphany 2013

Riemannian Geometry IV, Homework 7 (Week 18)

Due date for starred problems: **Tuesday, March 12**.

7.1. (a) Let $c(t)$ be a geodesic, and $c(t_0)$ is conjugate to $c(t_1)$. Let J be any Jacobi field along c vanishing at t_0 and t_1. Show that J is orthogonal, i.e. $\langle J(t), c'(t)\rangle \equiv 0$.

(b) Show that the dimension of the space $J_c^\perp$ of orthogonal vector fields along c is $2n-2$.

Solution:

(a) We have proved in class that the function $t \to \langle J(t), c'(t)\rangle$ is linear. Since it is equal to zero at two points t_0 and t_1, it vanishes everywhere, so $J(t)$ is orthogonal.

(b) Recall that $J(t)$ is orthogonal if and only if both $\langle J(0), c'(0)\rangle$ and $\langle \nabla_t J(0), c'(0)\rangle$ vanish. Each of these equations defines a codimension one subspace in M_p, so the dimension of $J_c^\perp = (n-1)+(n-1) = 2n-2$.

7.2. ($\star$) Let $c : [0,1] \to M$ be a geodesic, and J be a Jacobi field along c. Denote $c(0)=p, c'(0)=v$. Define a curve $\gamma(s)$,

$$\gamma : (-\varepsilon, \varepsilon) \to M, \qquad \gamma(0) = p, \gamma'(0) = J(0)$$

Define also a vector field $V(s) \in \Gamma(\gamma^{-1}TM)$, such that

$$V(0) = v, \qquad \nabla_s V(0) = \nabla_t J(0),$$

and a variation $F(s,t) = \exp_{\gamma(s)} tV(s)$.

(a) Show that $F(s,t)$ is a geodesic variation of $c(t)$.

(b) Show that $\frac{\partial F}{\partial s}(0,0) = \gamma'(0) = J(0)$, and $\nabla_t \frac{\partial F}{\partial s}(0,0) = \nabla_s V(0) = \nabla_t J(0)$.

(c) Deduce from (a) and (b) that every Jacobi field along a geodesic $c(t)$ is a variational vector field of some geodesic variation of c.

Solution:

(a) By definition of exponential map, for given s the curve $t \to \exp_{\gamma(s)} tV(s)$ is geodesic.

(b) We have

$$\frac{\partial F}{\partial s}(0,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(s)} tV(s)\Big|_{t=0} = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(s)}(0) = \frac{\partial}{\partial s}\Big|_{s=0} \gamma(s) = \gamma'(0) = J(0),$$

and

$$\nabla_t \frac{\partial F}{\partial s}(0,0) = \nabla_s\Big|_{s=0} \frac{\partial F}{\partial t}\Big|_{t=0}(s,t) = \nabla_s\Big|_{s=0} V(s) = \nabla_s V(0) = \nabla_t J(0)$$

(c) According to (a), the variation F is geodesic, thus its variational vector field $\frac{\partial F}{\partial s}(0,t)$ is Jacobi. By (b), $\frac{\partial F}{\partial s}(0,0) = J(0)$, and $\nabla_t \frac{\partial F}{\partial s}(0,0) = \nabla_t J(0)$, which means that $\frac{\partial F}{\partial s}(0,t) = J(t)$ due to uniqueness theorem.

7.3. Let $c : [a, b] \to M$ be a geodesic. For $t \in [a, b]$, denote by J_c^t the space of Jacobi fields along c vanishing at t. Let $a \leq t_0 < t_1 \leq b$, and define a map $\psi : J_c^{t_0} \to M_{c(t_1)}$ by $\psi(J) = J(t_1)$.

(a) Show that ψ is linear.

Now assume that $c(t_0)$ is not conjugate to $c(t_1)$.

(b) Show that ψ is an isomorphism.

(c) Show that for any $u_1 \in M_{c(t_1)}$ there exists a unique $J \in J_c^{t_0}$ such that $J(t_1) = u_1$.

(d) Using the arguments similar to ones from (a)–(c), show that for any $u_0 \in M_{c(t_0)}$ there exists a unique $J \in J_c^{t_1}$ such that $J(t_0) = u_0$.

(e) Deduce from (c) and (d) that if $c(t_0)$ is not conjugate to $c(t_1)$, then for any $u_0 \in M_{c(t_0)}$ and $u_1 \in M_{c(t_1)}$ there exists a unique Jacobi field J along C such that $J(t_0) = u_0$ and $J(t_1) = u_1$.

Solution:

(a) $\psi(J_1 + J_2) = (J_1 + J_2)(t_1) = J_1(t_1) + J_2(t_1) = \psi(J_1) + \psi(J_2)$.

(b) If the kernel of ψ contains a non-zero vector, then $c(t_1)$ is conjugate to $c(t_0)$, which is not the case. Thus, ψ is a monomorphism. Note that the dimensions of both $J_c^{t_0}$ and $M_{c(t_1)}$ are equal to n. Therefore, ψ is a monomorphism of vector spaces of the same dimension, which implies that it is an isomorphism.

(c),(d) This follows directly from (b).

(e) Let $J_1 \in J_c^{t_0}$ such that $J_1(t_1) = u_1$, and $J_2 \in J_c^{t_1}$ such that $J_2(t_0) = u_0$. Then for $J = J_1 + J_2$ we have $J(t_0) = u_0$ and $J(t_1) = u_1$.

Assume there is $J' \neq J$ with the same properties. Then for $J_1' = J' - J_2$ we have $J_1'(t_0) = 0$ and $J_1'(t_1) = u_1$, so $J_1' = J_1$. Hence, $J' = J$.