Durham University
Pavel Tumarkin

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Riemannian Geometry IV, Solutions 1 (Week 11)

1.1. ($\star$) Consider the upper half-plane $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$$

(a) Show that all the Christoffel symbols are zero except $\Gamma^2_{22} = -\frac{1}{2y}$.

(b) Show that the vertical segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ is a geodesic curve when parametrized proportionally to arc length.

(c) Show that the length of the segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ tends to 2 as ε tends to zero.

(d) Show that (M, g) is not geodesically complete.

Solution:

(a) We use the formula

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{m=1}^{n} g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m})$$

The only non-zero $g_{ij,k}$ is $g_{22,2} = -1/y^2$. Thus, the only non-zero Christoffel symbol is

$$\Gamma^2_{22} = \frac{1}{2} g^{22}(g_{22,2}) = -\frac{1}{2y}$$

(b) <u>Solution 1.</u> Parametrize the segment by $c(t) = (0, \alpha(t))$, where $\alpha(0) = \varepsilon, \alpha(1) = 1$, and $\alpha(t)$ is increasing. Then $c'(t) = \alpha'(t)\frac{\partial}{\partial y}$, and we obtain

$$\|c'(t)\| = |\alpha'(t)|\|\frac{\partial}{\partial y}\| = \frac{\alpha'(t)}{\sqrt{y}} = \frac{\alpha'(t)}{\sqrt{\alpha(t)}}$$

Since we want $c(t)$ to be parametrized proportionally to arc length, we have

$$\|c'(t)\| = \frac{\alpha'(t)}{\sqrt{\alpha(t)}} = k$$

for some $k \in \mathbb{R}$, so

$$(*) \qquad \alpha'(t) = k\sqrt{\alpha(t)}.$$

To show that $c(t)$ is geodesic, we need to show that $\frac{D}{dt}c'(t) = 0$, where $\frac{D}{dt}$ denotes covariant derivative along $c(t)$. Computing, we obtain

$$\frac{D}{dt}c'(t) = \frac{D}{dt}\left(\alpha'(t)\frac{\partial}{\partial y}\right) = \alpha''(t)\frac{\partial}{\partial y} + \alpha'(t)\frac{D}{dt}\frac{\partial}{\partial y} =$$

$$= \alpha''(t)\frac{\partial}{\partial y} + \alpha'(t)\nabla_{\alpha'(t)\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \alpha''(t)\frac{\partial}{\partial y} + {\alpha'}^2(t)\left(-\frac{1}{2y}\frac{\partial}{\partial y}\right) = \left(\alpha''(t) - \frac{{\alpha'}^2(t)}{2\alpha(t)}\right)\frac{\partial}{\partial y}$$

Applying $(*)$, we obtain $\alpha''(t) = k^2/2$, and ${\alpha'}^2(t)/2\alpha(t) = k^2/2$ as well, so $\frac{D}{dt}c'(t) = 0$.

Solution 2. (based on symmetry and uniqueness).

Consider the map $R : M \to M$, $R(x, y) = (-x, y)$ (reflection with respect to the y-axis). As the metric g_{ij} depends on y only (which is preserved by R), R is an isometry. (Indeed, the differential of this map is the diagonal matrix $DR = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so $DR(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x}$ and $DR(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}$. Hence, $\langle DR(v), DR(w)\rangle = \langle v, w\rangle$ for any $v, w \in T_{(x,y)}M$.) Thus, R takes each geodesic to a geodesic.

Now, let $\gamma(t)$ be the geodesic such that $\gamma(0) = (0, 1)$, $\gamma'(0) = (0, -1)$. Suppose that γ does not belong entirely to the vertical line, i.e. for some t_0 the point $\gamma(t_0)$ has non-zero y-coordinate (say, positive). Then the geodesic $R(\gamma(t))$ obtained from γ by the reflection R does not coincide with $\gamma(t)$ (it has strictly negative y-coordinate at t_0) and satisfies the same initial conditions as $\gamma(t)$. This contradicts the uniqueness of a geodesic starting from a given point in a given direction.

(c)

$$\int_0^1 \frac{\alpha'(t)}{\sqrt{\alpha(t)}}\,dt = \int_0^1 2\left(\sqrt{\alpha(t)}\right)'\,dt = 2\sqrt{\alpha(1)} - 2\sqrt{\alpha(0)} = 2 - 2\varepsilon$$

which tends to 2 as ε tends to zero.

(d) It follows from (c) that the sequence $1/n$ is a Cauchy sequence, but does not converge in M. Thus, (M, g) is not complete, and by the Hopf – Rinow theorem it is not geodesically complete.

1.2. ($\star$) Let $H_3(\mathbb{R})$ be the set of 3×3 unit upper-triangular matrices (i.e. the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, x_3 \in \mathbb{R}$).

(a) Show that $H_3(\mathbb{R})$ is a group with respect to matrix multiplication. This group is called the *Heisenberg group*.

(b) Show that the Heisenberg group is a Lie group. What is its dimension?

(c) Prove that the matrices

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of the tangent space $T_eH_3(\mathbb{R})$ of the group $H_3(\mathbb{R})$ at the neutral element e.

(d) For each $k = 1, 2, 3$, find an explicit formula for the curve $c_k : \mathbb{R} \to H_3(\mathbb{R})$ given by $c_k(t) = \mathrm{Exp}\,(tX_k)$.

Solution:

(a) It is an easy computation to check the axioms of a group (i.e H_3 is closed under multiplication, there exists an obvious neutral element (3×3 identity matrix), there is an inverse element for each $h \in H_3$, associativity works as always in matrix groups).

(b) The matrix elements (x_1, x_2, x_3) give a global chart on H_3, so H_3 is a smooth 3-manifold. The multiplication g_1g_2 can be written as $(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2 + x_1y_3, x_3 + y_3)$, and the inverse element g_1^{-1} can be written as $(x_1, x_2, x_3)^{-1} = (-x_1, x_1x_3 - x_2, -x_3)$, which are smooth maps $H_3 \times H_3 \to H_3$ and $H_3 \to H_3$ respectively. Hence, H_3 is a Lie group.

(c) To see that the matrices X_i belong to T_eH_3 consider the paths $c_i(t) = I + X_it \in H_3$. By definition, $\frac{\partial}{\partial x_i} = c_i'(t) = X_i$. So, $\{X_1, X_2, X_3\}$ is the basis of T_eH_3 since $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$ is a basis.

(d) Since $X_i^2 = 0$ for $i = 1, 2, 3$ we see that $\mathrm{Exp}\,(tX_i) = I + X_it$.

1.3. Let G, H be Lie groups. A map $\varphi : G \to H$ is called a *homomorphism (of Lie groups)* if it is smooth and it is a homomorphism of abstract groups.

Denote by $\mathfrak{g}, \mathfrak{h}$ Lie algebras of G and H, and let $\varphi : G \to H$ be a homomorphism.

(a) Show that the differential $D\varphi(e) : T_eG \to T_eH$ induces a linear map $D\varphi : \mathfrak{g} \to \mathfrak{h}$, where $D\varphi(X)$ for $X \in \mathfrak{g}$ is the unique left-invariant vector field on H such that $D\varphi(X)(e) = D\varphi(X(e))$.

(b) Show that for any $g \in G$

$$L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$$

(c) Show that for any $X \in \mathfrak{g}$ and $g \in G$

$$D\varphi(X)(\varphi(g)) = D\varphi(X(g))$$

(d) Show that $D\varphi : \mathfrak{g} \to \mathfrak{h}$ is a *homomorphism of Lie algebras*, i.e. a linear map satisfying $D\varphi([X, Y]) = [D\varphi(X), D\varphi(Y)]$ for any $X, Y \in \mathfrak{g}$.

Solution:

(a) The map $D\varphi : \mathfrak{g} \to \mathfrak{h}$ defined by $D\varphi(X)(e) = D\varphi(X(e))$ is clearly linear.

(b) Since φ is a homomorphism, we have for $h \in G$

$$(L_{\varphi(g)} \circ \varphi)(h) = \varphi(g)\varphi(h) = \varphi(gh) = \varphi(L_g(h)) = \varphi \circ L_g(h)$$

(c) Since $D\varphi(X) \in \mathfrak{h}$, we have

$$\begin{aligned} D\varphi(X)(\varphi(g)) = DL_{\varphi(g)}(e)D\varphi(X)(e) = DL_{\varphi(g)}(e)D\varphi(X(e)) = D(L_{\varphi(g)} \circ \varphi)(e)X(e) = \\ = D(\varphi \circ L_g)X(e) = D\varphi(DL_gX(e)) = D\varphi(X(g)) \end{aligned}$$

(d) Reproducing the proof of Prop. 6.8 (substituting L_g by φ and making use of (c) and Lemma 6.7), we have for every $f \in C^\infty(H)$ and $g \in G$

$$\begin{aligned} (D\varphi \circ [X,Y](g))(f) = [X,Y](g)(f \circ \varphi) &= X(g)Y(f \circ \varphi) - Y(g)X(f \circ \varphi) = \\ &= X(g)((D\varphi \circ Y)(f)) - Y(g)((D\varphi \circ X)(f)) = \\ &= X(g)(D\varphi(Y)(f) \circ \varphi) - Y(g)(D\varphi(X)(f) \circ \varphi) = \\ &= D\varphi(X(g))(D\varphi(Y)(f)) - D\varphi(Y(g))(D\varphi(X)(f)) = \\ &= D\varphi(X)(\varphi(g))(D\varphi(Y)(f)) - D\varphi(Y)(\varphi(g))(D\varphi(X)(f)) = \\ &= [D\varphi(X), D\varphi(Y)](\varphi(g))(f) \end{aligned}$$

In particular, taking $g = e$, we have $(D\varphi \circ [X, Y])(e) = [D\varphi(X), D\varphi(Y)](e)$. According to (c), we have $D\varphi([X, Y]) \circ \varphi = D\varphi \circ [X, Y]$, so $(D\varphi \circ [X, Y])(e) = D\varphi([X, Y])(e)$. Therefore, we have two left-invariant vector fields $D\varphi([X, Y])$ and $[D\varphi(X), D\varphi(Y)]$ coinciding at e, which implies they are equal.