Durham University
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Epiphany 2015

Riemannian Geometry IV, Homework 1 (Week 11)

Due date for starred problems: **Wednesday, January 28**.

1.1. ($\star$) Consider the upper half-plane $M = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$$

(a) Show that all the Christoffel symbols are zero except $\Gamma^2_{22} = -\frac{1}{2y}$.
(b) Show that the vertical segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ is a geodesic curve when parametrized proportionally to arc length.
(c) Show that the length of the segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ tends to 2 as ε tends to zero.
(d) Show that (M, g) is not geodesically complete.

1.2. ($\star$) Let $H_3(\mathbb{R})$ be the set of 3×3 unit upper-triangular matrices (i.e. the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, x_3 \in \mathbb{R}$).

(a) Show that $H_3(\mathbb{R})$ is a group with respect to matrix multiplication. This group is called the *Heisenberg group*.
(b) Show that the Heisenberg group is a Lie group. What is its dimension?
(c) Prove that the matrices

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of the tangent space $T_e H_3(\mathbb{R})$ of the group $H_3(\mathbb{R})$ at the neutral element e.
(d) For each $k = 1, 2, 3$, find an explicit formula for the curve $c_k : \mathbb{R} \to H_3(\mathbb{R})$ given by $c_k(t) = \mathrm{Exp}\,(tX_k)$.

1.3. Let G, H be Lie groups. A map $\varphi : G \to H$ is called a *homomorphism (of Lie groups)* if it is smooth and it is a homomorphism of abstract groups.

Denote by $\mathfrak{g}, \mathfrak{h}$ Lie algebras of G and H, and let $\varphi : G \to H$ be a homomorphism.

(a) Show that the differential $D\varphi(e) : T_eG \to T_eH$ induces a linear map $D\varphi : \mathfrak{g} \to \mathfrak{h}$, where $D\varphi(X)$ for $X \in \mathfrak{g}$ is the unique left-invariant vector field on H such that $D\varphi(X)(e) = D\varphi(X(e))$.
(b) Show that for any $g \in G$

$$L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$$

(c) Show that for any $X \in \mathfrak{g}$ and $g \in G$

$$D\varphi(X)(\varphi(g)) = D\varphi(X(g))$$

(d) Show that $D\varphi : \mathfrak{g} \to \mathfrak{h}$ is a *homomorphism of Lie algebras*, i.e. a linear map satisfying $D\varphi([X,Y]) = [D\varphi(X), D\varphi(Y)]$ for any $X, Y \in \mathfrak{g}$.