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Riemannian Geometry IV, Solutions 2 (Week 12)

2.1. Let $G \subset GL_n(\mathbb{R})$, $v, w \in T_I G$. Use the definition

$$\mathrm{ad}_w v = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \mathrm{Exp}\,(tw)\,\mathrm{Exp}\,(sv)\,\mathrm{Exp}\,(-tw)$$

of the adjoint representation and the expansion of the power series for exponents of tw and sv to show that $\mathrm{ad}_w v = [w, v]$.

Solution: This can be done by a straightforward computation. Namely, by expanding all the exponents as power series and collecting the coefficients of $t^1 s^1$ in the product one can immediately see that the coefficient is $wv - vw$. Now observe that after taking derivatives with respect to s and t at $(0, 0)$ one obtains exactly the coefficient of $t^1 s^1$.

2.2. (a) Let $A, B \in M_n(\mathbb{R})$, $[A, B] = 0$. Take $t \in \mathbb{R}$ and show that $\mathrm{Exp}\,(t(A + B)) = \mathrm{Exp}\,(tA)\,\mathrm{Exp}\,(tB)$ (in particular, you obtain that $\mathrm{Exp}\,(A + B) = \mathrm{Exp}\,(A)\,\mathrm{Exp}\,(B)$).

(b) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\right) = \begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Guess what would be the exponential of an $n \times n$-matrix of the same form (i.e., a Jordan block with zero eigenvalue).

(c) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}c&1&0&0\\0&c&1&0\\0&0&c&1\\0&0&0&c\end{pmatrix}\right) = e^{tc}\begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Solution:

(a) As in the previous exercise, expand both exponents $\mathrm{Exp}\,(tA)$ and $\mathrm{Exp}\,(tB)$ as power series and collect the coefficient of t^n in the product. The monomials involved will be of type $\frac{(tA)^k (tB)^{n-k}}{k!(n-k)!}$, so the monomial containing t^n in the product will be

$$\sum_{k=0}^{n} \frac{(tA)^k (tB)^{n-k}}{k!(n-k)!} = \sum_{k=0}^{n} t^n \frac{A^k B^{n-k}}{k!(n-k)!} = \frac{t^n}{n!} \sum_{k=0}^{n} A^k B^{n-k} \frac{n!}{k!(n-k)!} = \frac{t^n}{n!}(A + B)^n$$

(b) Let $A = \begin{pmatrix}0&t&0&0\\0&0&t&0\\0&0&0&t\\0&0&0&0\end{pmatrix}$. We have

$$A^2 = \begin{pmatrix}0&0&t^2&0\\0&0&0&t^2\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad A^3 = \begin{pmatrix}0&0&0&t^3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad A^k = 0 \qquad \text{for all } k \geq aspowerseriesand4.$$

So the power series $\mathrm{Exp}\,(A)$ terminates after 4 terms and we conclude that

$$\mathrm{Exp}\,(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 = \begin{pmatrix}1&t&t^2/2&t^3/(3!)\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

(c) Let $B = tcI$, where I denotes the 4×4 identity matrix, and let A be as in (a). Then we have $\mathrm{Exp}\,(B) = e^{tc}I$ and A and B commute. This implies that

$$\mathrm{Exp}\left(t\begin{pmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix}\right) = \mathrm{Exp}\,(A+B) = \mathrm{Exp}\,(B)\mathrm{Exp}\,(A) = e^{tc}\begin{pmatrix} 1 & t & t^2/2 & t^3/(3!) \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.3. ($\star$) Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a *bi-invariant* Riemannian metric (i.e., both L_g and R_g are isometries for every $g \in G$). Let $\mathfrak{g}$ denote the Lie algebra of G, and let $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\langle X, Y \rangle$ is a constant function on G.

(b) Use the relation

$$\langle Z, \nabla_X Y \rangle = \frac{1}{2}\left(X\langle Z, Y\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle + \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle - \langle Z, [Y, X]\rangle \right)$$

and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y\rangle = \langle Y, [Z, Y]\rangle$.

(c) By Corollary 6.18, the bi-invariance of the metric implies that

$$\langle [U, X], V\rangle = -\langle U, [V, X]\rangle$$

for $X, U, V \in \mathfrak{g}$. Use this fact to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(d) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

Solution:

(a)

$$\langle X(g), Y(g)\rangle_g = \langle DL_g(e)X(e), DL_g(e)Y(e)\rangle_g = \langle X(e), Y(e)\rangle_e,$$

so $\langle X(g), Y(g)\rangle_g$ does not depend on g.

(b) The relation with 6 terms in the RHS implies that

$$\langle Z, \nabla_Y Y\rangle = \frac{1}{2}\left(Y\langle Z, Y\rangle + Y\langle Z, Y\rangle - Z\langle Y, Y\rangle + \langle Y, [Z, Y]\rangle + \langle Y, [Z, Y]\rangle - \langle Z, [Y, Y]\rangle \right) = \frac{1}{2}\left(\langle Y, [Z, Y]\rangle + \langle Y, [Z, Y]\rangle \right),$$

since the first three derivatives of the right hand side of the relation vanish by (a). Moreover, we have $[Y, Y] = 0$. Thus, we conclude that

$$\langle Z, \nabla_Y Y\rangle = \langle Y, [Z, Y]\rangle.$$

(c) The bi-invariance implies that

$$\langle [Y, X], Y\rangle = -\langle Y, [Y, X]\rangle = -\langle [Y, X], Y\rangle,$$

so $\langle [Y, X], Y\rangle = 0$. This gives us $\langle X, \nabla_Y Y\rangle = 0$ for all left-invariant X, so we have $\nabla_Y Y = 0$ for all left-invariant Y.

(d) We calculate

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y = \nabla_X Y + \nabla_Y X = 2\nabla_X Y - [X, Y].$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

2.4. Let G be a Lie group, $H \subset G$ be a closed subgroup, $\pi : G \to G/H$ be the canonical projection. Let $\langle \cdot, \cdot \rangle_e$ be an Ad_H-*invariant inner product* on T_eG (i.e. $\langle \mathrm{Ad}_h v, \mathrm{Ad}_h w \rangle_e = \langle v, w \rangle_e$ for every $h \in H$, $v, w \in T_eG$). Define $V \subset T_eG$ to be the orthogonal complement to $T_eH \subset T_eG$ with respect to $\langle \cdot, \cdot \rangle_e$, and let Φ be the restriction of $D\pi(e) : T_eG \to T_{eH}\, G/H$ to the subspace V. Prove the following statements:

(a) $T_eH = \ker D\pi(e)$.
(You may use without proof that $D\pi(e) : T_eG \to T_{eH}\, G/H$ is surjective.)

(b) $\Phi : V \to T_{eH}\, G/H$ is an isomorphism.

(c) V is Ad_H-invariant.

Solution:

Let $N = \dim G$ and $n = \dim H$.

(a) We first show that $T_eH \subset \ker D\pi(e)$. Let $v \in T_eH$. Then there exists a curve $c : (-\varepsilon, \varepsilon) \to H$ such that $c(0) = e$ and $c'(0) = v$. The image curve $\pi \circ c : (-\varepsilon, \varepsilon) \to G/H$ is constant because of $c(t)H = eH$ for all $t \in (-\varepsilon, \varepsilon)$. This implies that

$$D\pi(e)(v) = \frac{d}{dt}\bigg|_{t=0} \pi \circ c(t) = 0 \in T_{eH}G/H.$$

$D\pi(e) : T_eG \to T_{eH}G/H$ is surjective, and we have by the dimension formula:

$$\dim \ker D\pi(e) + \dim T_{eH}G/H = \dim T_eG,$$

i.e., $\dim \ker D\pi(e) = N - (N - n) = n$. Since $\dim T_eH = n$, we conclude that $T_eH = \ker D\pi(e)$.

(b) Note first that $\dim V = \dim T_eG - \dim \ker D\pi(e) = N - n$ and $\dim T_{eH}G/H = N - n$, so we are done if we prove that Φ is surjective (then it is also injective by dimension argument). We know that $D\pi(e) : T_eG \to T_{eH}G/H$ is surjective. For a given $v \in T_{eH}G/H$ let $v_1 \in T_eG$ such that $D\pi(e)(v_1) = v$. Let $v_1 = u_1 + w_1 \in T_eH \perp V$. Since $T_eH = \ker D\pi(e)$, we have $v = D\pi(e)(v_1) = D\pi(e)(w_1) = \Phi(w_1)$. This shows surjectivity of Φ.

(c) We first show that T_eH is Ad_H-invariant. Let $v \in T_eH = \ker D\pi(e)$. Then there is a curve $c : (-\epsilon, \epsilon) \to H$ such that $c(0) = e$ and $c'(0) = v$, and we have

$$D\pi(e)(Ad_h v) = \frac{d}{dt}\bigg|_{t=0} \pi(\underbrace{hc(t)h^{-1}}_{\in H}) = 0 \in T_{eH}G/H,$$

i.e., $Ad_h v \in \ker D\pi(e) = T_eH$. Recall that $\langle \cdot, \cdot \rangle_e$ is Ad_H-invariant. Let $v \in V$. We need to show that $Ad_h v \perp T_eH$. Let $h \in H$ and $w \in T_eH$. Then

$$\langle Ad_h v, w \rangle_e = \langle Ad_{h^{-1}} Ad_h v, Ad_{h^{-1}} w \rangle_e = \langle \underbrace{v}_{\in V}, \underbrace{Ad_{h^{-1}} w}_{\in T_eH} \rangle_e = 0.$$

Here we used $Ad_{h_1} Ad_{h_2} = Ad_{h_1 h_2}$ (check this!)