Durham University
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Epiphany 2015

Riemannian Geometry IV, Homework 2 (Week 12)

Due date for starred problems: **Wednesday, January 28**.

2.1. Let $G \subset GL_n(\mathbb{R})$, $v, w \in T_I G$. Use the definition

$$\mathrm{ad}_w v = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \mathrm{Exp}\,(tw)\,\mathrm{Exp}\,(sv)\,\mathrm{Exp}\,(-tw)$$

of the adjoint representation and the expansion of the power series for exponents of tw and sv to show that $\mathrm{ad}_w v = [w, v]$.

2.2. (a) Let $A, B \in M_n(\mathbb{R})$, $[A, B] = 0$. Take $t \in \mathbb{R}$ and show that $\mathrm{Exp}\,(t(A+B)) = \mathrm{Exp}\,(tA)\,\mathrm{Exp}\,(tB)$ (in particular, you obtain that $\mathrm{Exp}\,(A+B) = \mathrm{Exp}\,(A)\,\mathrm{Exp}\,(B)$).

(b) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\right)=\begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Guess what would be the exponential of an $n \times n$-matrix of the same form (i.e., a Jordan block with zero eigenvalue).

(c) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}c&1&0&0\\0&c&1&0\\0&0&c&1\\0&0&0&c\end{pmatrix}\right)=e^{tc}\begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

2.3. ($\star$) Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with a *bi-invariant* Riemannian metric (i.e., both L_g and R_g are isometries for every $g \in G$). Let $\mathfrak{g}$ denote the Lie algebra of G, and let $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\langle X, Y\rangle$ is a constant function on G.

(b) Use the relation

$$\langle Z, \nabla_X Y\rangle = \frac{1}{2}\left(X\langle Z, Y\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle + \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle - \langle Z, [Y, X]\rangle\right)$$

and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y\rangle = \langle Y, [Z, Y]\rangle$.

(c) By Corollary 6.18, the bi-invariance of the metric implies that

$$\langle [U, X], V\rangle = -\langle U, [V, X]\rangle$$

for $X, U, V \in \mathfrak{g}$. Use this fact to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(d) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

2.4. Let G be a Lie group, $H \subset G$ be a closed subgroup, $\pi : G \to G/H$ be the canonical projection. Let $\langle\cdot,\cdot\rangle_e$ be an Ad_H-*invariant inner product* on T_eG (i.e. $\langle \mathrm{Ad}_h v, \mathrm{Ad}_h w\rangle_e = \langle v, w\rangle_e$ for every $h \in H$, $v, w \in T_eG$). Define $V \subset T_eG$ to be the orthogonal complement to $T_eH \subset T_eG$ with respect to $\langle\cdot,\cdot\rangle_e$, and let Φ the restriction of $D\pi(e) : T_eG \to T_{eH}\,G/H$ to the subspace V. Prove the following statements:

(a) $T_eH = \ker D\pi(e)$.
(You may use without proof that $D\pi(e) : T_eG \to T_{eH}\,G/H$ is surjective.)

(b) $\Phi : V \to T_{eH}\,G/H$ is an isomorphism.

(c) V is Ad_H-invariant.