Durham University
Pavel Tumarkin

Epiphany 2015

Riemannian Geometry IV, Solutions 3 (Week 13)

3.1. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;
(b) $R(X, fY)Z = fR(X, Y)Z$;
(c) $\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle$;
(d) $R(fX, gY)hZ = fghR(X, Y)Z$.

Solution:

(a) Note that $[fX, Y] = f[X, Y] - (Yf)X$. We have

$$\begin{aligned}R(fX, Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z = \\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-(Yf)X}Z = \\ &= f\nabla_X\nabla_Y Z - (Yf)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z + (Yf)\nabla_X Z = \\ &= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) = fR(X, Y)Z.\end{aligned}$$

(b) Using the symmetry $R(X, Y)Z = -R(Y, X)Z$ and applying (a) we obtain

$$R(X, fY)Z = -R(fY, X)Z = -fR(Y, X)Z = fR(X, Y)Z.$$

(c) Using the symmetry $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$ twice, we obtain

$$\begin{aligned}\langle R(X, Y)fZ, W\rangle &= \langle R(fZ, W)X, Y\rangle = \langle fR(Z, W)X, Y\rangle = \\ &= f\langle R(Z, W)X, Y\rangle = f\langle R(X, Y)Z, W\rangle = \langle fR(X, Y)Z, W\rangle.\end{aligned}$$

(d) Since (c) holds for all vector fields W, we conclude that

$$R(X, Y)fZ = fR(X, Y)Z.$$

Using this together with (a) and (b), we obtain

$$R(fX, gY)hZ = fghR(X, Y)Z.$$

3.2. ($\star$) **First Bianchi Identity**
Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

for X, Y, Z vector fields on M by reducing the equation to *Jacobi identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Solution: We have

$$\begin{aligned}&R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = \\ &= (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) + (\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X) + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y) = \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) - (\nabla_{[X,Y]}Z) + \nabla_{[Y,Z]}X) + \nabla_{[Z,X]}Y) = \\ &= \nabla_X[Y, Z] + \nabla_Y[Z, X] + \nabla_Z[X, Y] - (\nabla_{[X,Y]}Z) + \nabla_{[Y,Z]}X) + \nabla_{[Z,X]}Y) = \\ &= (\nabla_X[Y, Z] - \nabla_{[Y,Z]}X) + (\nabla_Y[Z, X] - \nabla_{[Z,X]}Y) + (\nabla_Z[X, Y] - \nabla_{[X,Y]}Z) = \\ &= -([[Y, Z], X] + [[Z, X], Y] + [[X, Y], Z]) = 0.\end{aligned}$$

3.3. ($\star$) Parametrize the sphere S_r^2 of radius r in $\mathbb{R}^3$ by

$$(x, y, z) = (r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta),$$

and consider the metric on S_r^2 induced by the Euclidean metric in $\mathbb{R}^3$.

(a) Compute $R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})$.

(b) Compute the sectional curvature of S_r^2.

Solution:

(a) First, we compute Christoffel symbols.
Since

$$\frac{\partial}{\partial\varphi} = (-r\sin\varphi\sin\vartheta, r\cos\varphi\sin\vartheta, 0),$$

$$\frac{\partial}{\partial\vartheta} = (r\cos\varphi\cos\vartheta, r\sin\varphi\cos\vartheta, -r\sin\vartheta),$$

we have $(g_{ij}) = \begin{pmatrix} r^2\sin^2\vartheta & 0 \\ 0 & r^2 \end{pmatrix}$ and $(g^{ij}) = \begin{pmatrix} \frac{1}{r^2\sin^2\vartheta} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$.
So, $g_{11,2} = 2r^2\sin\vartheta\cos\vartheta$ and $g_{ij,k} = 0$ for all other choices of i, j, k. Since the metric is diagonal, we have $\Gamma_{ij}^k = \frac{1}{2}g^{kk}(g_{ik,j} + g_{kj,i} - g_{ij,k})$ which is non-zero only if the (unordered) triple (i, j, k) coincides with $(1, 1, 2)$. This implies

$$\Gamma_{11}^2 = \frac{1}{2}\frac{1}{r^2}(-2r^2\sin\vartheta\cos\vartheta) = -\sin\vartheta\cos\vartheta,$$

$$\Gamma_{12}^1 = \frac{1}{2}\frac{1}{r^2\sin^2\vartheta}(2r^2\sin\vartheta\cos\vartheta) = \cot\vartheta.$$

By definition of Christoffel symbols,

$$\nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\varphi} = \nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\vartheta} = \cot\vartheta\frac{\partial}{\partial\varphi},$$

$$\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = -sin\vartheta\cos\vartheta\frac{\partial}{\partial\vartheta} \qquad \text{and} \qquad \nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\vartheta} = 0.$$

$\nabla_{[\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}]}X = 0$ for any X since $[\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}] = 0$.
Now we compute the Riemann curvature tensor.

$$R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}) = \langle R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle =$$

$$= \langle \nabla_{\frac{\partial}{\partial\varphi}}\nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\varphi} - \nabla_{\frac{\partial}{\partial\vartheta}}\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} - \nabla_{[\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}]}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle =$$

$$= \langle \nabla_{\frac{\partial}{\partial\varphi}}(\cot\vartheta)\frac{\partial}{\partial\varphi} - \nabla_{\frac{\partial}{\partial\vartheta}}(-\sin\vartheta\cos\vartheta\frac{\partial}{\partial\vartheta}), \frac{\partial}{\partial\vartheta}\rangle =$$

$$= \langle(\frac{\partial}{\partial\varphi}\cot\vartheta)\frac{\partial}{\partial\vartheta} + \cot\vartheta\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} + (\frac{\partial}{\partial\vartheta}\sin\vartheta\cos\vartheta)\frac{\partial}{\partial\vartheta} + \sin\vartheta\cos\vartheta\nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}\rangle =$$

$$= \langle \cot\vartheta(-\sin\vartheta\cos\vartheta)\frac{\partial}{\partial\vartheta} + (\cos^2\vartheta - \sin^2\vartheta)\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}\rangle =$$

$$= -r^2\cos^2\vartheta + r^2(\cos^2\vartheta - \sin^2\vartheta) = -r^2\sin^2\vartheta.$$

(b)

$$K = \frac{\langle R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}\rangle}{\|\frac{\partial}{\partial\varphi}\|^2\|\frac{\partial}{\partial\vartheta}\|^2 - \langle\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle} = \frac{r^2\sin^2\vartheta}{r^2\sin^2\vartheta \cdot r^2} = \frac{1}{r^2}.$$

3.4. Let (M, g) be a Riemannian manifold. The goal of this exercise is to show that M is of constant sectional curvature K_0 if and only if

$$\langle R(v_1, v_2)v_3, v_4\rangle = -K_0(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

for any $p \in M$ and $v_1, v_2, v_3, v_4 \in T_pM$. Denote the expression $-K_0\left(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_2, v_3\rangle\langle v_1, v_4\rangle\right)$ by (v_1, v_2, v_3, v_4).

(a) Show that if

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$, then M is of constant sectional curvature K_0.

Now assume that M is of constant sectional curvature K_0. Our aim is to show that

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$.

(b) Show that the expression (v_1, v_2, v_3, v_4) is a tensor, i.e. it is multilinear.

(c) Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemann curvature tensor has. Namely,
- $(v_1, v_2, v_3, v_4) = -(v_2, v_1, v_3, v_4)$
- $(v_1, v_2, v_3, v_4) = -(v_1, v_2, v_4, v_3)$
- $(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$
- $(v_1, v_2, v_3, v_4) + (v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0$

(d) Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(e) Show that if no more than three distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(f) Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$

i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

(g) Use Bianchi identity to prove the initial statement.

Solution:

(a) If the equality holds, we have

$$K(v, u) = \frac{\langle R(v, u)u, v\rangle}{\|v\|^2\|u\|^2 - \langle v, u\rangle^2} = \frac{K_0(\langle v, v\rangle\langle u, u\rangle - \langle v, u\rangle\langle u, v\rangle)}{\|v\|^2\|u\|^2 - \langle v, u\rangle^2} = K_0$$

(b) This can be seen from the explicit formula for (v_1, v_2, v_3, v_4).

(c) Straighforward calculations using the definition of (v_1, v_2, v_3, v_4).

(d) By the definition of sectional curvature,

$$\langle R(v, u)u, v\rangle = K_0\left(\|v\|^2\|u\|^2 - \langle v, u\rangle^2\right) = (v, u, u, v).$$

For collections of vectors ordered in other way the statement follows by (c).

(e) Using linearity and (d), we obtain

$$\begin{aligned}\langle R(v_1, v_2 + v_3)v_2 + v_3, v_1\rangle &= (v_1, v_2 + v_3, v_2 + v_3, v_1) = \\ &= (v_1, v_2, v_2, v_1) + (v_1, v_2, v_3, v_1) + (v_1, v_3, v_2, v_1) + (v_1, v_3, v_3, v_1),\end{aligned}$$

and on the other side

$$\begin{aligned}&\langle R(v_1, v_2 + v_3)v_2 + v_3, v_1\rangle = \\ &= \langle R(v_1, v_2)v_2, v_1\rangle + \langle R(v_1, v_2)v_3, v_1\rangle + \langle R(v_1, v_3)v_2, v_1\rangle + \langle R(v_1, v_3)v_3, v_1\rangle = \\ &= (v_1, v_2, v_2, v_1) + \langle R(v_1, v_2)v_3, v_1\rangle + \langle R(v_1, v_3)v_2, v_1\rangle + (v_1, v_3, v_3, v_1),\end{aligned}$$

which leads to

$$\langle R(v_1,v_2)v_3,v_1\rangle + \langle R(v_1,v_3)v_2,v_1\rangle = (v_1,v_2,v_3,v_1) + (v_1,v_3,v_2,v_1). \tag{1}$$

By the symmetries, we obtain

$$\langle R(v_1,v_2)v_3,v_1\rangle = \langle R(v_1,v_3)v_2,v_1\rangle,$$

and the same holds for $(\cdot,\cdot,\cdot,\cdot)$, so (1) simplifies to

$$2\langle R(v_1,v_2)v_3,v_1\rangle = 2(v_1,v_2,v_3,v_1).$$

(f) Using (e), we obtain on one side

$$\begin{aligned}\langle R(v_1+v_4,v_2)v_3,v_1+v_4\rangle &= (v_1+v_4,v_2,v_3,v_1+v_4) = \\ &= (v_1,v_2,v_3,v_1)+(v_1,v_2,v_3,v_4)+(v_4,v_2,v_3,v_1)+(v_4,v_2,v_3,v_4),\end{aligned}$$

and on the other side

$$\begin{aligned}&\langle R(v_1+v_4,v_2)v_3,v_1+v_4\rangle = \\ &= \langle R(v_1,v_2)v_3,v_1\rangle + \langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_4,v_2)v_3,v_1\rangle + \langle R(v_4,v_2)v_3,v_4\rangle = \\ &= (v_1,v_2,v_3,v_1) + \langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_4,v_2)v_3,v_1\rangle + (v_4,v_2,v_3,v_4).\end{aligned}$$

Comparing both expressions, we conclude that

$$\langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_4,v_2)v_3,v_1\rangle = (v_1,v_2,v_3,v_4) + (v_4,v_2,v_3,v_1).$$

This implies

$$\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4) = -\langle R(v_4,v_2)v_3,v_1\rangle + (v_4,v_2,v_3,v_1).$$

Using the symmetries, we derive

$$-\langle R(v_4,v_2)v_3,v_1\rangle = -\langle R(v_3,v_1)v_4,v_2\rangle = \langle R(v_3,v_1)v_2,v_4\rangle,$$

and the same identity for $(\cdot,\cdot,\cdot,\cdot)$, so we end up with

$$\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4) = \langle R(v_3,v_1)v_2,v_4\rangle - (v_3,v_1,v_2,v_4)$$

(g) Using (f) and Bianchi identity, we conclude that

$$\begin{aligned}&3\left(\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4)\right) = \left(\langle R(v_1,v_2)v_3,v_4\rangle - (v_1,v_2,v_3,v_4)\right) + \\ &+ \left(\langle R(v_3,v_1)v_2,v_4\rangle - (v_3,v_1,v_2,v_4)\right) + \left(\langle R(v_3,v_3)v_1,v_4\rangle - (v_2,v_3,v_1,v_4)\right) = \\ &= \left(\langle R(v_1,v_2)v_3,v_4\rangle + \langle R(v_3,v_1)v_2,v_4\rangle + \langle R(v_2,v_3)v_1,v_4\rangle\right) - \\ &- \left((v_1,v_2,v_3,v_4) + (v_3,v_1,v_2,v_4) + (v_2,v_3,v_1,v_4)\right) = 0 - 0 = 0,\end{aligned}$$

which completes the proof.

3.5. A Riemannian manifold (M,g) is called *Einstein manifold* if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v,w) = c\langle v,w\rangle$$

for every $p \in M$, $v, w \in T_pM$.

(a) Show that (M,g) is Einstein manifold if and only if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v) = c$$

for every $p \in M$ and unit tangent vector $v \in T_pM$.

(b) Show that if (M,g) is of constant sectional curvature then (M,g) is Einstein manifold.

Solution: We have seen in class that $Ric_p(v,w)$ is a symmetric bilinear form on T_pM, and thus $Ric_p(v)$ is a quadratic form.

(a) If M is Einstein manifold, then

$$Ric_p(v) = Ric_p(v,v) = c\langle v, v\rangle,$$

which is equal to c for any unit vector v.

Conversely, if $Ric_p(v) = c$ for any unit vector v, then, by linearity,

$$Ric_p(\lambda v) = c\lambda^2 = c\langle \lambda v, \lambda v\rangle,$$

which implies

$$Ric_p(u) = c\langle u, u\rangle$$

for arbitrary vector $u \in T_pM$. Now, reconstructing symmetric bilinear form $Ric_p(v,w)$ by quadratic form $Ric_p(v) = Ric_p(v,v)$, we obtain

$$Ric_p(v,w) = \frac{1}{2}(Ric_p(v+w, v+w) - Ric_p(v) - Ric_p(w)) =$$

$$= \frac{1}{2}(c\langle v+w, v+w\rangle - c\langle v, v\rangle - c\langle w, w\rangle) = c\langle v, w\rangle$$

(b) Let M be n-dimensional, $p \in M$, and assume $K(\Pi) = K_0$ for all 2-dimensional subspaces Π of TM. Take arbitrary unit vector $v \in T_pM$, extend it to an orthonormal basis $\{v, v_2, \ldots, v_n\}$. Then

$$Ric_p(v) = \sum_{i=2}^{n} K(v, v_i) = (n-1)K_0,$$

so M is Einstein manifold.