Durham University
Pavel Tumarkin Epiphany 2015

Riemannian Geometry IV, Solutions 6 (Week 16)

6.1. (a) Let $c(t)$ be a geodesic, and let $c(t_0)$ be conjugate to $c(t_1)$. Let J be any Jacobi field along c vanishing at t_0 and t_1. Show that J is orthogonal, i.e. $\langle J(t), c'(t)\rangle \equiv 0$.

(b) Show that the dimension of the space $J_c^\perp$ of orthogonal vector fields along c is $2n-2$.

Solution:

(a) We have proved in class that the function $t \mapsto \langle J(t), c'(t)\rangle$ is linear. Since it is equal to zero at two points t_0 and t_1, it vanishes everywhere, so $J(t)$ is orthogonal.

(b) Recall that $J(t)$ is orthogonal if and only if both $\langle J(0), c'(0)\rangle$ and $\langle \frac{D}{dt}J(0), c'(0)\rangle$ vanish. Each of these equations defines a codimension one subspace in T_pM, so the dimension of $J_c^\perp = (n-1)+(n-1) = 2n-2$.

6.2. ($\star$) Let $c : [0,1] \to M$ be a geodesic, and let J be a Jacobi field along c. Denote $c(0) = p, c'(0) = v$. Define a curve $\gamma(s)$,

$$\gamma : (-\varepsilon, \varepsilon) \to M, \qquad \gamma(0) = p, \gamma'(0) = J(0)$$

Define also a vector field $V(s) \in \mathfrak{X}_\gamma(M)$, such that

$$V(0) = v, \qquad \frac{D}{ds}V(0) = \frac{D}{dt}J(0),$$

and a variation $F(s,t) = exp_{\gamma(s)}tV(s)$.

(a) Show that $F(s,t)$ is a geodesic variation of $c(t)$.

(b) Show that $\frac{\partial F}{\partial s}(0,0) = \gamma'(0) = J(0)$, and $\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{ds}V(0) = \frac{D}{dt}J(0)$.

(c) Deduce from (a) and (b) that every Jacobi field along a geodesic $c(t)$ is a variational vector field of some geodesic variation of c.

Solution:

(a) By the definition of the exponential map, for given s the curve $t \mapsto \exp_{\gamma(s)} tV(s)$ is a geodesic.

(b) We have

$$\frac{\partial F}{\partial s}(0,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(s)} tV(s)\Big|_{t=0} = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(s)}(0) = \frac{\partial}{\partial s}\Big|_{s=0}\gamma(s) = \gamma'(0) = J(0),$$

and

$$\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial t}\Big|_{t=0}(s,t) = \frac{D}{ds}\Big|_{s=0}V(s) = \frac{D}{ds}V(0) = \frac{D}{dt}J(0)$$

(c) According to (a), the variation F is geodesic, thus its variational vector field $\frac{\partial F}{\partial s}(0,t)$ is Jacobi. By (b), $\frac{\partial F}{\partial s}(0,0) = J(0)$, and $\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{dt}J(0)$, which means that $\frac{\partial F}{\partial s}(0,t) = J(t)$ due to the uniqueness theorem.

6.3. Jacobi fields and conjugate points on locally symmetric spaces
A Riemannian manifold (M,g) is called a *locally symmetric space* if $\nabla R = 0$ (see Exercise 9.3). Let (M,g) be an n-dimensional locally symmetric space and $c : [0,\infty) \to M$ be a geodesic with $p = c(0)$ and $v = c'(0) \in T_pM$. Prove the following facts:

(a) Let X, Y, Z be parallel vector fields along c. Show that $R(X,Y)Z$ is also parallel.

(b) Let $K_v \in \mathrm{Hom}\,(T_pM, T_pM)$ be the curvature operator defined by

$$K_v(w) = R(w, v)v.$$

Show that K_v is self-adjoint, i.e.,

$$\langle K_v(w_1), w_2\rangle = \langle w_1, K_v(w_2)\rangle$$

for every pair of vectors $w_1, w_2 \in T_pM$.

(c) Choose an orthonormal basis $w_1, \ldots, w_n \in T_pM$ that diagonalizes K_v, i.e.,

$$K_v(w_i) = \lambda_i w_i$$

(such a basis exists since K_v is self-adjoint). Let $W_1, \ldots, W_n$ be the parallel vector fields along c with $W_i(0) = w_i$ (i.e., $\{W_i\}$ form a parallel orthonormal basis along c). Show that for all $t \in [0, \infty)$

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).$$

(d) Let $J(t) = \sum_i J_i(t)W_i(t)$ be a Jacobi field along c. Show that Jacobi equation translates into

$$J_i''(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \ldots, n.$$

(e) Show that the conjugate points of p along c are given by $c(\pi k/\sqrt{\lambda_i})$, where k is any positive integer and λ_i is a positive eigenvalue of K_v.

Solution:

(a) We know that $\nabla R = 0$. Let $\frac{D}{dt}$ denote covariant derivative along c. Then we have, for parallel vector fields X, Y, Z along c that

$$\begin{aligned}0 = \nabla R(X, Y, Z, c')(t) = \frac{D}{dt} R(X(t), Y(t))Z(t) - \\ - R(\underbrace{\frac{D}{dt}X(t)}_{=0}, Y(t))Z(t) - R(X(t), \underbrace{\frac{D}{dt}Y(t)}_{=0})Z(t) - R(X(t), Y(t))\underbrace{\frac{D}{dt}Z(t)}_{=0} = \\ = \frac{D}{dt}R(X(t), Y(t))Z(t).\end{aligned}$$

This shows that $R(X, Y)Z$ is parallel.

(b) The symmetries of R yield

$$\langle K_v(w_1), w_2\rangle = \langle R(w_1, v)v, w_2\rangle = \langle R(w_2, v)v, w_1\rangle = \langle K_v(w_2), w_1\rangle = \langle w_1, K_v(w_2)\rangle.$$

(c) Since K_v is self-adjoint, we can find an orthonormal basis $w_1, \ldots, w_n \in T_pM$ with $K_v(w_i) = \lambda_i w_i$. We know, by (a), that $K_{c'(t)}(W_i(t)) = R(c'(t), W_i(t))W_i(t)$ is parallel and, since $K_{c'(0)}(W_i(0)) = K_v(w_i) = \lambda_i w_i$, we must have

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t),$$

since parallel vector fields V along c are uniquely determined by their initial values $V(0) \in T_pM$.

(d) Let J be a Jacobi field along c. Then J satisfies the Jacobi equation

$$\frac{D^2}{dt^2}J + R(J, c')c' = 0.$$

Since $W_1, \ldots, W_n$ form a parallel basis along c, we obtain, by taking inner product with W_i:

$$\begin{aligned}0 = \langle \frac{D^2}{dt^2}J, W_i\rangle + \langle R(J, c')c', W_i\rangle = \\ = \frac{d^2}{dt^2}\sum_j J_j\langle W_j, W_i\rangle + \sum_j J_j\langle R(W_j, c')c', W_i\rangle = \\ = J_i'' + \sum_j J_j\lambda_j\langle W_j, W_i\rangle = J_i'' + \lambda_i J_i.\end{aligned}$$

(e) The unique solution of $J_i''(t) + \lambda_i J_i(t) = 0$, $J_i(0) = 0$ (up to scalar multiples) is given by

$$J_i(t) = \begin{cases} \sin(t\sqrt{\lambda_i}) & \text{if } \lambda_i > 0, \\ t & \text{if } \lambda_i = 0, \\ \sinh(t\sqrt{-\lambda_i}) & \text{if } \lambda_i < 0. \end{cases}$$

So J_i has zeros for positive t only if $\lambda_i > 0$, and these are precisely at $t = \pi k/\sqrt{\lambda_i}$. The corresponding Jacobi fields with $J(0) = 0$ and $\frac{D}{dt}J(0) = w_i$ produce the conjugate points $c(\pi k/\sqrt{\lambda_i})$.