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Riemannian Geometry IV, Term 1 (Sections 3–4)

3 Riemannian metric

Definition 3.1. Let M be a smooth manifold. A Riemannian metric $g_p(\cdot,\cdot)$ or $\langle\cdot,\cdot\rangle_p$ is a family of real inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ depending smoothly on $p \in M$. A smooth manifold M with a Riemannian metric g is called a Riemannian manifold (M, g).

Examples 3.2–3.3. Euclidean metric on $\mathbb{R}^n$, induced metric on $M \subset \mathbb{R}^n$.

Definition 3.4. Let (M, g) be a Riemannian manifold. For $v \in T_pM$ define the length of v by $0 \le \|v\|_g = \sqrt{g_p(v, v)}$.

Example 3.5. Three models of hyperbolic geometry:

model	notation	M	g
Hyperboloid	$\mathbb{W}^n$	$\{y \in \mathbb{R}^{n+1} \mid q(y,y) = -1, y_{n+1} > 0\}$ where $q(x,y) = \sum_{i=1}^n x_iy_i - x_{n+1}y_{n+1}$	$g_x(v,w) = q(v,w)$
Poincaré ball	$\mathbb{B}^n$	$\{x \in \mathbb{R}^n \mid \|x\|^2 = \sum\limits_{i=1}^n x_i^2 < 1\}$	$g_x(v,w) = \frac{4}{(1-\|x\|^2)^2}\langle v,w\rangle$
Upper half-space	$\mathbb{H}^n$	$\{x \in \mathbb{R}^n \mid x_n > 0\}$	$g_x(v,w) = \frac{1}{x_n^2}\langle v,w\rangle$

Definition 3.6. Given two vector spaces V_1, V_2 with real inner products $(V_i, \langle\cdot,\cdot\rangle_i)$, an isomorphism $T : V_1 \to V_2$ of vector spaces is a linear isometry if $\langle Tv, Tw\rangle_2 = \langle v, w\rangle_1$ for all $v, w \in V_1$.

This is equivalent to preserving the lengths of all vectors (since $\langle v, w\rangle = \frac{1}{2}(\langle v + w, v + w\rangle - \langle v, v\rangle - \langle w, w\rangle)$).

Definition 3.7. A diffeomorphism $f : (M, g) \to (N, h)$ of two Riemannian manifolds is an isometry if $Df(p) : T_pM \to T_{f(p)}N$ is a linear isometry for all $p \in M$.

Theorem 3.8 (Nash embedding theorem)**.** *For any Riemannian manifold (M^m, g) the exists an isometric embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$. If M is compact, there exists such $k \le \frac{m(3m+1)}{2}$, and if M is not compact, there is such $k \le \frac{m(m+1)(3m+1)}{2}$.*

Definition 3.9. (M, g) is a Riemannian manifold, $c : [a, b] \to M$ is a smooth curve. The length $L(c)$ of c is defined by $L(c) = \int_a^b \|c'(t)\|dt$, where $\|c'(t)\| = \langle c'(t), c'(t)\rangle_{c(t)}^{1/2}$. The length of a piecewise-smooth curve is defined as the sum of lengths of its smooth pieces.

Theorem 3.10 (Reparametrization)**.** *Let $\varphi : [c, d] \to [a, b]$ be a strictly monotonic smooth function, $\varphi' \ne 0$, and let $\gamma : [a, b] \to M$ be a smooth curve. Then for $\tilde\gamma = \gamma \circ \varphi : [c, d] \to M$ holds $L(\gamma) = L(\tilde\gamma)$.*

Definition 3.11. A smooth curve $c : [a, b] \to M$ is arc-length parametrized if $\|c'(t)\| \equiv 1$.

Proposition 3.12 (evident)**.** *If a curve $c : [a, b] \to M$ is arc-length parametrized, then $L(c) = b - a$.*

Proposition 3.13. *Every curve has an arc-length parametrization.*

Example 3.14. Length of vertical segments in $\mathbb{H}$. Shortest paths between points on vertical rays.

Definition 3.15. Define a distance $d : M \times M \to [0, \infty)$ on (M, g) by $d(p, q) = \inf_\gamma\{L(\gamma)\}$, where γ is a piecewise smooth curve connecting p and q.

Remark. (M, d) is a metric space.

Example 3.16. Induced metric on $S^1 \subset \mathbb{R}^2$.

Definition 3.17. If (M, g) is a Riemannian manifold, then any subset $A \subset M$ is also a metric space with the induced metric $d|_{A\times A} : A \times A \to [0, \infty)$ defined by $d(p, q) = \inf_\gamma\{L(\gamma) \mid \gamma : [a, b] \to A, \gamma(a) = p, \gamma(b) = q\}$, where the length $L(\gamma)$ is computed in M.

Example 3.18. Punctured Riemann sphere: $\mathbb{R}^n$ with metric $g_x(v, w) = \frac{4}{(1+\|x\|^2)^2}\langle v, w\rangle$.

4 Levi-Civita connection and parallel transport

4.1 Levi-Civita connection

Example 4.1. Given a vector field $X = \sum a_i(p)\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$ and a vector $v \in T_p\mathbb{R}^n$ define the covariant derivative of X in direction v in $\mathbb{R}^n$ by $\nabla_v(X) = \lim\limits_{t\to 0} \frac{X(p+tv)-X(p)}{t} = \sum v(a_i)\frac{\partial}{\partial x_i}\Big|_p \in T_p\mathbb{R}^n$.

Proposition 4.2. *The covariant derivative $\nabla_v X$ in $\mathbb{R}^n$ satisfies all the properties (a)–(e) listed below in Definition 4.3 and Theorem 4.4.*

Definition 4.3. Let M be a smooth manifold. A map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$ is affine connection if for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ holds

(a) $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$

(b) $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$

(c) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

Theorem 4.4 (Levi-Civita, Fundamental Theorem of Riemannian Geometry)**.** *Let (M, g) be a Riemannian manifold. There exists a unique affine connection ∇ on M with the additional properties for all $X, Y, Z \in \mathfrak{X}(M)$:*

(d) $v(\langle X, Y\rangle) = \langle\nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$ *(Riemannian property);*

(e) $\nabla_X Y - \nabla_Y X = [X, Y]$ *(∇ is torsion-free).*

This connection is called Levi-Civita connection of (M, g).

Remark 4.5. Properties of Levi-Civita connection in $\mathbb{R}^n$ and in $M \subset \mathbb{R}^n$ with induced metric.

4.2 Christoffel symbols

Definition 4.6. Let ∇ be the Levi-Civita connection on (M, g), and let $\varphi : U \to V$ be a coordinate chart with coordinates $\varphi = (x_1, \ldots, x_n)$. Since $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}(p) \in T_pM$, there exists a uniquely determined collection of functions $\Gamma^k_{ij} \in C^\infty(U)$ s.t. $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}(p) = \sum_{k=1}^n \Gamma^k_{ij}(p)\frac{\partial}{\partial x_k}(p)$. These functions are called Christoffel symbols of ∇ with respect to the chart φ.

Remark. Christoffel symbols determine ∇ since $\nabla_{\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}} \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j,k} a_i b_j \Gamma_{ij}^k \frac{\partial}{\partial x_k}$.

Proposition 4.7.

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km} (g_{im,j} + g_{jm,i} - g_{ij,m}),$$

where $g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab}$ *and* $(g^{ij}) = (g_{ij})^{-1}$, *i.e.* $\{g^{ij}\}$ *are the elements of the matrix inverse to* (g_{ij}).

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Example 4.8. In $\mathbb{R}^n$, $\Gamma_{ij}^k \equiv 0$ for all i, j, k. Computation of Γ_{ij}^k in $S^2 \subset \mathbb{R}^3$ with induced metric.

4.3 Parallel transport

Definition 4.9. Let $c : (a, b) \to M$ be a smooth curve. A smooth map $X : (a, b) \to TM$ with $X(t) \in T_{c(t)}M$ is called a vector field along c. These fields form a vector space $\mathfrak{X}_c(M)$.

Example 4.10. $c'(t) \in \mathfrak{X}_c(M)$.

Proposition 4.11. *Let* (M, g) *be a Riemannian manifold, let* ∇ *be the Levi-Civita connection,* $c : (a, b) \to M$ *be a smooth curve. There exists a unique map* $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ *satisfying*

(a) $\frac{D}{dt}(\alpha X + Y) = \alpha \frac{D}{dt} X + \frac{D}{dt} Y$ *for any* $\alpha \in \mathbb{R}$.

(b) $\frac{D}{dt}(fX) = f'(t)X + f \frac{D}{dt} X$ *for every* $f \in C^\infty(M)$.

(c) If $\widetilde{X} \in \mathfrak{X}(M)$ *is a local extension of* X
(i.e. there exists $t_0 \in (a, b)$ *and* $\varepsilon > 0$ *such that* $X(t) = \widetilde{X}\big|_{c(t)}$ *for all* $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$*)*
then $(\frac{D}{dt} X)(t_0) = \nabla_{c'(t_0)} \widetilde{X}$.

This map $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ *is called the covariant derivative along the curve c.*

Example 4.12. Covariant derivative in $\mathbb{R}^n$.

Definition 4.13. Let $X \in \mathfrak{X}_c(M)$. If $\frac{D}{dt} X = 0$ then X is said to be parallel along c.

Example 4.14. A vector field X in $\mathbb{R}^n$ is parallel along a curve if and only if X is constant.

Theorem 4.15. *Let* $c : [a, b] \to M$ *be a smooth curve,* $v \in T_{c(a)}M$. *There exists a unique vector field* $X \in \mathfrak{X}_c(M)$ *parallel along* c *with* $X(a) = v$.

Corollary 4.16. *Parallel vector fields form a vector space of dimension* n *(where* n *is the dimension of* (M, g)*).*

Definition 4.17. Let $c : [a, b] \to M$ be a smooth curve. A linear map $P_c : T_{c(a)}M \to T_{c(b)}M$ defined by $P_c(v) = X(b)$, where $X \in \mathfrak{X}_c(M)$ is parallel along c with $X(a) = v$, is called a parallel transport along c.

Remark. The parallel transport P_c depends on the curve c (not only on its endpoints).

Proposition 4.18. *The parallel transport* $P_c : T_{c(a)}M \to T_{c(b)}M$ *is a linear isometry between* $T_{c(a)}M$ *and* $T_{c(b)}M$, *i.e.* $g_{c(a)}(v, w) = g_{c(b)}(P_c v, P_c w)$.