

Riemannian Geometry IV, Term 2 (Section 6)

6 Crash course: Basics about Lie groups

6.1 Left-invariant vector fields and Lie algebra

Definition 6.1. A Lie group G is a smooth manifold with a smooth group structure, i.e. the maps $G \times G \rightarrow G$, $(g, h) \mapsto gh$ and $G \rightarrow G$, $g \mapsto g^{-1}$ are smooth.

Examples. Matrix Lie groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$.

Definition 6.2. Let G be a Lie group, $g \in G$. Then the maps $L_g : G \rightarrow G$ and $R_g : G \rightarrow G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ are called left- and right-translation. L_g and R_g are diffeomorphisms of G .

Remark. (a) $L_{g^{-1}} \circ L_g = id_G$, $L_{g_1} R_{g_2}(h) = R_{g_2} L_{g_1}(h) = g_1 h g_2$.

(b) The differential $DL_g : T_h G \rightarrow T_{gh} G$ gives a natural identification of tangent spaces.

Example 6.3. Let $G \subset GL_n(\mathbb{R})$ be a matrix group, $v \in T_e G$. Then $DL_g(e)v = gv$.

Definition 6.4. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if for any $g \in G$ $DL_g X = X \circ L_g$, i.e. $DL_g(h)X(h) = X(gh)$.

Remark 6.5. (a) Left-invariant vector fields on G form a vector space over \mathbb{R} .

(b) Left-invariant vector field is determined by its value at e : $X(g) = DL_g(e)X(e)$.

(c) Hence, the space of left-invariant vector fields on G can be identified with $T_e G$.

Definition 6.6. The space of left-invariant vector fields on G is called the Lie algebra of G and denoted by \mathfrak{g} .

Lemma 6.7. Let M, N be smooth manifolds, $X \in \mathfrak{X}(M)$, $f \in C^\infty(N)$, $p \in M$, and let $\varphi : M \rightarrow N$ be a smooth map. Then

$$(d\varphi(p)X(p))f = X(p)(f \circ \varphi)$$

Proposition 6.8. Let X be a Lie group with Lie algebra \mathfrak{g} . Then for any $X, Y \in \mathfrak{g}$ the Lie bracket $[X, Y] \in \mathfrak{g}$. Consequently, \mathfrak{g} is indeed a Lie algebra (see Definition 2.22).

6.2 Lie group exponential map and adjoint representation

Definition 6.9. Define $\text{Exp} : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ by $\text{Exp}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

Properties. (a) The infinite sum converges for any matrix $A \in M_n(\mathbb{R})$, so $\text{Exp}(A)$ is well-defined;

(b) $\text{Exp}(0) = I$;

(c) if $AB = BA$ then $\text{Exp}(A + B) = \text{Exp}(A) \cdot \text{Exp}(B)$; in particular, $\text{Exp}(-A)\text{Exp}(A) = I$, so $\text{Exp}(A) \in GL_n(\mathbb{R})$ for any $A \in M_n(\mathbb{R})$.

Proposition 6.10. *Let G be a matrix Lie group. Let $v \in T_e G$ and let X be the unique left-invariant vector field on G with $X(e) = v$. Then the curve $c(t) = \text{Exp}(tv) \in G$ satisfies $c(0) = e$, $c'(0) = v$ and $c'(t) = X(c(t))$.*

A curve of the form $c(t) = \text{Exp}(tv)$ is called a 1-parameter subgroup of G with $c'(0) = v$.

Remark. For an abstract Lie group the exponential map can be defined as follows. Let G be a Lie group and \mathfrak{g} be its Lie algebra. Let $v \in T_e G$ and let $X \in \mathfrak{g}$ be the unique left-invariant vector field with $X(e) = v$. Then there exists a unique curve $c_v : \mathbb{R} \rightarrow G$ with $c_v(0) = e$, $c'_v(t) = X(c_v(t))$ [without proof]. The curve c_v is called an integral curve of X . We define the exponential map by $\text{Exp}(v) = c_v(1)$.

Example 6.11. Computation of the exponent for a diagonalizable matrix.

Definition 6.12. Let G be a Lie group. For $g \in G$ the adjoint representation $\text{Ad}_g : T_e G \rightarrow T_e G$ is defined by

$$\text{Ad}_g(w) = \left. \frac{d}{dt} \right|_{t=0} L_g R_{g^{-1}}(\text{Exp}(tw)) = \left. \frac{d}{dt} \right|_{t=0} g \text{Exp}(tw) g^{-1}.$$

For $v \in T_e G$ the adjoint representation $\text{ad}_v : T_e G \rightarrow T_e G$ is defined by

$$\text{ad}_v(w) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\text{Exp}(tv)}(w) = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \text{Exp}(tv) \text{Exp}(sw) \text{Exp}(-tv).$$

Theorem 6.13 (without proof). *Let G be a Lie group with a Lie algebra \mathfrak{g} . Then for all $X, Y \in \mathfrak{g}$ holds $\text{ad}_{X(e)} Y(e) = [X, Y](e) \in T_e G$, i.e. by canonical identification of \mathfrak{g} with $T_e G$ we have $\text{ad}_X Y = [X, Y]$.*

Example 6.14. Theorem 6.13 for the case of a matrix Lie group.

6.3 Riemannian metrics on Lie groups

Definition 6.15. For a given inner product $\langle \cdot, \cdot \rangle_e$ on $T_e G$, define the inner product at $g \in G$ for $v, w \in T_g G$ by $\langle v, w \rangle_g = \langle DL_{g^{-1}}(g)v, DL_{g^{-1}}(g)w \rangle_e$. The family $(\langle \cdot, \cdot \rangle_g)_{g \in G}$ of inner products defines a left-invariant Riemannian metric on G .

Remark 6.16. Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a left-invariant metric. Then

- (a) the diffeomorphisms $L_g : G \rightarrow G$ are isometries;
- (b) for any two left-invariant vector fields $X, Y \in \mathfrak{g}$ the function $g \mapsto \langle X(g), Y(g) \rangle_g$ is constant.

Theorem 6.17 (without proof). *Let G be a compact Lie group. Then G admits a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle_g$, i.e. both families of diffeomorphisms L_g and R_g are isometries.*

Corollary 6.18. *Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with bi-invariant metric, let $X, Y, Z \in \mathfrak{g}$. Then $\langle [X, Y], Z \rangle = -\langle [X, Z], Y \rangle$.*

Corollary 6.19. *Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with bi-invariant metric and let ∇ be the Levi-Civita connection. Then for $X, Y \in \mathfrak{g}$ holds $\nabla_X Y = \frac{1}{2}[X, Y]$.*

Corollary 6.20. (a) *1-parameter subgroups are exactly the geodesics of the bi-invariant metric on G ;*

- (b) *the Lie group exponential map Exp coincides with the Riemannian exponential map \exp_e at the identity.*

6.4 Homogeneous spaces

Definition 6.21. A connected Riemannian manifold (M, g) is called homogeneous if the group $Isom(M, g)$ of isometries of M acts transitively on M , i.e. for any $p, q \in M$ there exists $\varphi \in Isom(M, g)$ s.t. $\varphi(p) = q$.

Examples. \mathbb{E}^n , S^n , Lie groups.

Given a Lie group G and a closed subgroup $H \subset G$, consider the set $M = G/H = \{gH \mid g \in G\}$. Then M is a smooth manifold (non-trivial theorem, uses that H is closed).

Example. $G = SO_3(\mathbb{R})$, $H = SO_2(\mathbb{R}) \subset G$. Then $M = SO_3/SO_2 \approx S^2$. In general, $O_n(\mathbb{R})/O_{n-1}(\mathbb{R}) \approx S^{n-1}$.