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Riemannian Geometry IV, Solutions 2 (Week 12)

2.1. Let $G \subset GL_n(\mathbb{R})$, $v, w \in T_I G$. Use the definition

$$\mathrm{ad}_w v = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \operatorname{Exp}(tw)\operatorname{Exp}(sv)\operatorname{Exp}(-tw)$$

of the adjoint representation and the expansion of the power series for exponents of tw and sv to show that $\mathrm{ad}_w v = [w, v]$.

Solution: This can be done by a straightforward computation. Namely, by expanding all the exponents as power series and collecting the coefficients of $t^1 s^1$ in the product one can immediately see that the coefficient is $wv - vw$. Now observe that after taking derivatives with respect to s and t at $(0,0)$ one obtains exactly the coefficient of $t^1 s^1$.

2.2. (a) Let $A, B \in M_n(\mathbb{R})$, $[A, B] = 0$. Take $t \in \mathbb{R}$ and show that $\operatorname{Exp}(t(A+B)) = \operatorname{Exp}(tA)\operatorname{Exp}(tB)$ (in particular, you obtain that $\operatorname{Exp}(A+B) = \operatorname{Exp}(A)\operatorname{Exp}(B)$).

(b) Show that

$$\operatorname{Exp}\left(t\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\right) = \begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Guess what would be the exponential of an $n \times n$-matrix of the same form (i.e., a Jordan block with zero eigenvalue).

(c) Show that

$$\operatorname{Exp}\left(t\begin{pmatrix}c&1&0&0\\0&c&1&0\\0&0&c&1\\0&0&0&c\end{pmatrix}\right) = e^{tc}\begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Solution:

(a) As in the previous exercise, expand both exponents $\operatorname{Exp}(tA)$ and $\operatorname{Exp}(tB)$ as power series and collect the coefficient of t^n in the product. The monomials involved will be of type $\frac{(tA)^k(tB)^{n-k}}{k!(n-k)!}$, so the monomial containing t^n in the product will be

$$\sum_{k=0}^{n} \frac{(tA)^k(tB)^{n-k}}{k!(n-k)!} = \sum_{k=0}^{n} t^n \frac{A^k B^{n-k}}{k!(n-k)!} = \frac{t^n}{n!}\sum_{k=0}^{n} A^k B^{n-k}\frac{n!}{k!(n-k)!} = \frac{t^n}{n!}(A+B)^n$$

(b) Let $A = \begin{pmatrix}0&t&0&0\\0&0&t&0\\0&0&0&t\\0&0&0&0\end{pmatrix}$. We have

$$A^2 = \begin{pmatrix}0&0&t^2&0\\0&0&0&t^2\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad A^3 = \begin{pmatrix}0&0&0&t^3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad A^k = 0 \qquad \text{for all } k \geq 4.$$

So the power series $\operatorname{Exp}(A)$ terminates after 4 terms and we conclude that

$$\operatorname{Exp}(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 = \begin{pmatrix}1&t&t^2/2&t^3/(3!)\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

(c) Let $B = tcI$, where I denotes the 4×4 identity matrix, and let A be as in (a). Then we have $\mathrm{Exp}\,(B) = e^{tc}I$ and A and B commute. This implies that

$$\mathrm{Exp}\left(t\begin{pmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix}\right) = \mathrm{Exp}\,(A+B) = \mathrm{Exp}\,(B)\mathrm{Exp}\,(A) = e^{tc}\begin{pmatrix} 1 & t & t^2/2 & t^3/(3!) \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.3. ($\star$) Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a *bi-invariant* Riemannian metric (i.e., both L_g and R_g are isometries for every $g \in G$). Let $\mathfrak{g}$ denote the Lie algebra of G, and let $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\langle X, Y \rangle$ is a constant function on G.

(b) Use the relation

$$\langle Z, \nabla_X Y \rangle = \frac{1}{2}\left(X\langle Z, Y\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle + \langle X, [Z,Y]\rangle + \langle Y, [Z,X]\rangle - \langle Z, [Y,X]\rangle \right)$$

and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y\rangle = \langle Y, [Z,Y]\rangle$.

(c) By Corollary 6.18, the bi-invariance of the metric implies that

$$\langle [U,X], V\rangle = -\langle U, [V,X]\rangle$$

for $X, U, V \in \mathfrak{g}$. Use this fact to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(d) Show that $\nabla_X Y = \frac{1}{2}[X,Y]$.

Solution:

(a)
$$\langle X(g), Y(g)\rangle_g = \langle DL_g(e)X(e), DL_g(e)Y(e)\rangle_g = \langle X(e), Y(e)\rangle_e,$$

so $\langle X(g), Y(g)\rangle_g$ does not depend on g.

(b) The relation with 6 terms in the RHS implies that

$$\langle Z, \nabla_Y Y\rangle = \frac{1}{2}\left(Y\langle Z, Y\rangle + Y\langle Z, Y\rangle - Z\langle Y, Y\rangle + \langle Y, [Z,Y]\rangle + \langle Y, [Z,Y]\rangle - \langle Z, [Y,Y]\rangle\right) =$$
$$\frac{1}{2}\left(\langle Y, [Z,Y]\rangle + \langle Y, [Z,Y]\rangle\right),$$

since the first three derivatives of the right hand side of the relation vanish by (a). Moreover, we have $[Y,Y] = 0$. Thus, we conclude that

$$\langle Z, \nabla_Y Y\rangle = \langle Y, [Z,Y]\rangle.$$

(c) The bi-invariance implies that

$$\langle [Y,X], Y\rangle = -\langle Y, [Y,X]\rangle = -\langle [Y,X], Y\rangle,$$

so $\langle [Y,X], Y\rangle = 0$. This gives us $\langle X, \nabla_Y Y\rangle = 0$ for all left-invariant X, so we have $\nabla_Y Y = 0$ for all left-invariant Y.

(d) We calculate

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y = \nabla_X Y + \nabla_Y X = 2\nabla_X Y - [X,Y].$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

2.4. The *special unitary group* $SU_n \subset M_n(\mathbb{C})$ consists of $n \times n$ matrices A with complex entries and unit determinant satisfying the equation $\bar{A}^t A = I = A\bar{A}^t$.

(a) Show that SU_n forms a group under matrix multiplication.

(b) Show that SU_2 consists of all matrices of the form

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad z, w \in \mathbb{C}, \quad |z|^2 + |w|^2 = 1.$$

(c) Show that SU_2 is a smooth (real) manifold. Find its dimension.

(d) Show that SU_2 is a Lie group.

(e) Find the Lie algebra $\mathfrak{su}_2$ of SU_2 as a subspace of $M_2(\mathbb{C})$. Find any basis $\{v_1, v_2, v_3\}$ of $\mathfrak{su}_2$. Compute explicitly the left-invariant vector fields X_1, X_2, X_3 on SU_2 such that $X_i(I) = v_i$.

Solution:

(a) Let $A, B \in SU_n$. Then

$$(\overline{AB})^t(AB) = \bar{B}^t\bar{A}^tAB = \bar{B}^t(\bar{A}^tA)B = \bar{B}^tB = I,$$

so $AB \in SU_n$. Also, $\det \bar{A}^t \det A = \det I = 1$ and $\det \bar{A}^t = \overline{\det A}$, which implies $|\det A| = 1 \neq 0$. Thus, A^{-1} exists. Now observe that $(\bar{A}^t)^{-1}A^{-1} = (A\bar{A}^t)^{-1} = I$, so $A^{-1} \in SU_n$.

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$. Then, computing $\bar{A}^tA$, we see that $A \in SU_2$ if and only if the following equations hold:

$$|a|^2 + |b|^2 = 1, \quad |c|^2 + |d|^2 = 1, \quad a\bar{c} + b\bar{d} = 0, \quad ad - bc = 1.$$

Multiplying the last two equations by c and $\bar{d}$ respectively and adding them to each other, we see that $a(|c|^2 + |d|^2) = \bar{d}$, which implies $a = \bar{d}$. This, in its turn, immediately implies that $c = -\bar{b}$.

Thus, we proved that every $A \in SU_2$ has required form. Conversely, it is clear that every matrix of such form has unit determinant and satisfies $\bar{A}^tA = I$.

(c) We can embed SU_2 in $\mathbb{R}^4$ with coordinates $(x_1, \ldots, x_4)$ by writing $z = x_1 + ix_2$ and $w = x_3 + ix_4$. Thus, $SU_2 = f^{-1}(0)$ for $f : \mathbb{R}^4 \to \mathbb{R}$, $f(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1$. Since 0 is a regular value, SU_2 is a 3-dim smooth manifold (actually, the description above shows that SU_2 is the 3-dim sphere S^3).

(d) The multiplication and inverse are polynomials in the entries so they are clearly smooth.

(e) Let $A(t) = \begin{pmatrix} x_1(s) + ix_2(s) & x_3(s) + ix_4(s) \\ -x_3(s) + ix_4(s) & x_1(s) - ix_2(s) \end{pmatrix}$ be a curve in SU_2, $A(0) = I$. Differentiating the equation $x_1^2(s) + x_2^2(s) + x_3^2(s) + x_4^2(s) = 1$ at $s = 0$, we obtain $x_1'(0) = 0$. In other words,

$$\mathfrak{su}_2 = T_ISU_2 = \left\{ \begin{pmatrix} xi & w \\ -\bar{w} & -xi \end{pmatrix} \mid x \in \mathbb{R},\ w \in \mathbb{C},\ x^2 + |w|^2 = 1 \right\}.$$

We can take as a basis of $\mathfrak{su}_2$, for example, matrices

$$v_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

(this particular choice of signs can be explained by the fact that the matrices $\sigma_1 = iv_1, \sigma_2 = iv_2, \sigma_3 = iv_3$ are Pauli matrices you could meet in Quantum Mechanics).

To construct left-invariant fields X_i recall from Example 6.3 that for matrix groups $X_i(g) = gX_i(I)$. Thus, for $g = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$, we have

$$X_1(g) = \begin{pmatrix} -iw & -iz \\ -i\bar{z} & i\bar{w} \end{pmatrix}, \quad X_2(g) = \begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix}, \quad X_3(g) = \begin{pmatrix} -iz & iw \\ i\bar{w} & i\bar{z} \end{pmatrix}$$