Durham University
Pavel Tumarkin Epiphany 2016

Riemannian Geometry IV, Homework 2 (Week 12)

Due date for starred problems: **Thursday, February 4**.

2.1. Let $G \subset GL_n(\mathbb{R})$, $v, w \in T_I G$. Use the definition

$$\mathrm{ad}_w v = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \mathrm{Exp}\,(tw)\,\mathrm{Exp}\,(sv)\,\mathrm{Exp}\,(-tw)$$

of the adjoint representation and the expansion of the power series for exponents of tw and sv to show that $\mathrm{ad}_w v = [w, v]$.

2.2. (a) Let $A, B \in M_n(\mathbb{R})$, $[A, B] = 0$. Take $t \in \mathbb{R}$ and show that $\mathrm{Exp}\,(t(A+B)) = \mathrm{Exp}\,(tA)\,\mathrm{Exp}\,(tB)$ (in particular, you obtain that $\mathrm{Exp}\,(A+B) = \mathrm{Exp}\,(A)\,\mathrm{Exp}\,(B)$).

(b) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\right) = \begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Guess what would be the exponential of an $n \times n$-matrix of the same form (i.e., a Jordan block with zero eigenvalue).

(c) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}c&1&0&0\\0&c&1&0\\0&0&c&1\\0&0&0&c\end{pmatrix}\right) = e^{tc}\begin{pmatrix}1&t&t^2/2&t^3/6\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

2.3. ($\star$) Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with a *bi-invariant* Riemannian metric (i.e., both L_g and R_g are isometries for every $g \in G$). Let $\mathfrak{g}$ denote the Lie algebra of G, and let $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\langle X, Y\rangle$ is a constant function on G.

(b) Use the relation

$$\langle Z, \nabla_X Y\rangle = \frac{1}{2}\left(X\langle Z,Y\rangle + Y\langle Z,X\rangle - Z\langle Y,X\rangle + \langle X,[Z,Y]\rangle + \langle Y,[Z,X]\rangle - \langle Z,[Y,X]\rangle\right)$$

and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y\rangle = \langle Y, [Z, Y]\rangle$.

(c) By Corollary 6.18, the bi-invariance of the metric implies that

$$\langle [U, X], V\rangle = -\langle U, [V, X]\rangle$$

for $X, U, V \in \mathfrak{g}$. Use this fact to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(d) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

2.4. The *special unitary group* $SU_n \subset M_n(\mathbb{C})$ consists of $n \times n$ matrices A with complex entries and unit determinant satisfying the equation $\bar{A}^t A = I = A\bar{A}^t$.

(a) Show that SU_n forms a group under matrix multiplication.

(b) Show that SU_2 consists of all matrices of the form

$$\begin{pmatrix}z & w\\ -\bar{w} & \bar{z}\end{pmatrix}, \quad z, w \in \mathbb{C}, \quad |z|^2 + |w|^2 = 1.$$

(c) Show that SU_2 is a smooth (real) manifold. Find its dimension.

(d) Show that SU_2 is a Lie group.

(e) Find the Lie algebra $\mathfrak{su}_2$ of SU_2 as a subspace of $M_2(\mathbb{C})$. Find any basis $\{v_1, v_2, v_3\}$ of $\mathfrak{su}_2$. Compute explicitly the left-invariant vector fields X_1, X_2, X_3 on SU_2 such that $X_i(I) = v_i$.