Durham University
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Epiphany 2016

Riemannian Geometry IV, Homework 3 (Week 13)

Due date for starred problems: **Thursday, February 18**.

3.1. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;
(b) $R(X, fY)Z = fR(X, Y)Z$;
(c) $\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle$;
(d) $R(fX, gY)hZ = fghR(X, Y)Z$.

3.2. ($\star$) **First Bianchi Identity**
Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

for X, Y, Z vector fields on M by reducing the equation to *Jacobi identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

3.3. ($\star$) Parametrize the sphere S^2_r of radius r in $\mathbb{R}^3$ by

$$(x, y, z) = (r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta),$$

and consider the metric on S^2_r induced by the Euclidean metric in $\mathbb{R}^3$.

(a) Compute $R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})$.
(b) Compute the sectional curvature of S^2_r.

3.4. Let (M, g) be a Riemannian manifold. The goal of this exercise is to show that M is of constant sectional curvature K_0 if and only if

$$\langle R(v_1, v_2)v_3, v_4\rangle = -K_0(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

for any $p \in M$ and $v_1, v_2, v_3, v_4 \in T_pM$. Denote the expression $-K_0\left(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_2, v_3\rangle\langle v_1, v_4\rangle\right)$ by (v_1, v_2, v_3, v_4).

(a) Show that if

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$, then M is of constant sectional curvature K_0.

Now assume that M is of constant sectional curvature K_0. Our aim is to show that

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$.

(b) Show that the expression (v_1, v_2, v_3, v_4) is a tensor, i.e. it is multilinear.

(c) Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemann curvature tensor has. Namely,

- $(v_1, v_2, v_3, v_4) = -(v_2, v_1, v_3, v_4)$
- $(v_1, v_2, v_3, v_4) = -(v_1, v_2, v_4, v_3)$
- $(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$
- $(v_1, v_2, v_3, v_4) + (v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0$

(d) Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(e) Show that if no more than three distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(f) Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$

i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

(g) Use Bianchi identity to prove the initial statement.

3.5. A Riemannian manifold (M, g) is called *Einstein manifold* if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v, w) = c\langle v, w\rangle$$

for every $p \in M$, $v, w \in T_pM$.

(a) Show that (M, g) is Einstein manifold if and only if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v) = c$$

for every $p \in M$ and unit tangent vector $v \in T_pM$.

(b) Show that if (M, g) is of constant sectional curvature then (M, g) is Einstein manifold.