Durham University
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Riemannian Geometry IV, Solutions 4 (Week 14)

4.1. Constant sectional curvature of hyperbolic 3-space

Let $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ be the upper half-space model of the 3-dimensional hyperbolic space, i.e. its metric is defined by $g_{ij} = 0$ for $i \neq j$, $g_{ii} = 1/x_3^2$.

(a) Show that sectional curvatures $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$ and $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ in $\mathbb{H}^3$ are equal to -1.

(b) Use (a) and the linearity of the Riemann curvature tensor to show that for any $p \in \mathbb{H}^3$ and $v_1, v_2, v_3, v_4 \in T_p\mathbb{H}^3$

$$\langle R(v_1, v_2)v_3, v_4\rangle = -(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

holds.

(c) Use (b) to show that 3-dimensional hyperbolic space $\mathbb{H}^3$ has constant sectional curvature -1.

(d) Show that n-dimensional hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ with metric $g_{ij} = 0$ for $i \neq j$, $g_{ii} = 1/x_n^2$ has constant sectional curvature -1.

Solution:

(a) We can compute the Christoffel symbols in a standard way obtaining

$$\Gamma^3_{11} = \Gamma^3_{22} = \frac{1}{x_3}, \quad \Gamma^3_{33} = \Gamma^1_{13} = \Gamma^1_{31} = \Gamma^2_{23} = \Gamma^2_{32} = -\frac{1}{x_3},$$

the remaining ones are zero. Using this, we can also compute that

$$\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1} = \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2} = \frac{1}{x_3}\frac{\partial}{\partial x_3}, \quad \nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_3} = -\frac{1}{x_3}\frac{\partial}{\partial x_3}, \quad \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} = \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_1} = 0,$$
$$\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3} = \nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_1} = -\frac{1}{x_3}\frac{\partial}{\partial x_1}, \quad \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_3} = \nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_2} = -\frac{1}{x_3}\frac{\partial}{\partial x_2}.$$

Now, we compute $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$.

$$K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) = \frac{\left\langle R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right\rangle}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_2}\|^2 - \left\langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\rangle^2} =$$
$$= \frac{1}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_3}\|^2}\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2} - \nabla_{\frac{\partial}{\partial x_2}}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} - \nabla_{\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right]}\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right\rangle =$$
$$= x_3^2x_3^2\left\langle \nabla_{\frac{\partial}{\partial x_1}}\frac{1}{x_3}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1}\right\rangle = x_3^4\left\langle \frac{1}{x_3}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1}\right\rangle =$$
$$= x_3^4\left\langle -\frac{1}{x_3^2}\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right\rangle = -x_3^4\frac{1}{x_3^2}\frac{1}{x_3^2} = -1$$

and

$$K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}) = \frac{\left\langle R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1}\right\rangle}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_3}\|^2 - \left\langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\right\rangle^2} =$$
$$= \frac{1}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_3}\|^2}\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_3} - \nabla_{\frac{\partial}{\partial x_3}}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3} - \nabla_{\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\right]}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1}\right\rangle =$$
$$= x_3^2x_3^2\left\langle -\nabla_{\frac{\partial}{\partial x_1}}\frac{1}{x_3}\frac{\partial}{\partial x_3} + \nabla_{\frac{\partial}{\partial x_3}}\frac{1}{x_3}\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right\rangle = x_3^4\left\langle -\frac{1}{x_3}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_3}\frac{1}{x_3}\frac{\partial}{\partial x_1} + \frac{1}{x_3}\nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right\rangle =$$
$$= x_3^4\left\langle \frac{1}{x_3^2}\frac{\partial}{\partial x_1} - \frac{1}{x_3^2}\frac{\partial}{\partial x_1} + -\frac{1}{x_3^2}\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right\rangle = -x_3^4\frac{1}{x_3^2}\frac{1}{x_3^2} = -1$$

Computation of $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is similar.

Remark. In fact, the plane spanned by vectors $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}$ is tangent to vertical hyperbolic plane $x_2 = c$, so the corresponding sectional curvature is exactly the curvature of hyperbolic plane which is equal to -1. Thus, we could avoid the computation of $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$.

(b) By computations similar to ones done in (a), we obtain that

$$\left\langle R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right\rangle = \left\langle R(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} \right\rangle = \left\langle R(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1})\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle = 0$$

Now we see that for all vectors $\{v_1, v_2, v_3, v_4\} \subset \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$ we have an equality

$$\langle R(v_1, v_2)v_3, v_4\rangle = -\left(\langle v_1, v_4\rangle \langle v_2, v_3\rangle - \langle v_1, v_3\rangle \langle v_2, v_4\rangle\right)$$

By linearity, the equality above holds for any quadruple of tangent vectors.

(c) This follows from (b) and Exercise 3.4.

(d) It is easy to see that the Christoffel symbol Γ_{ij}^k is not zero if and only if one of (i, j, k) equals n and two others are equal. This implies that if all of (i, j, k, l) are distinct then R_{ijkl} vanishes. Applying the arguments of (b) we conclude that $\mathbb{H}^n$ has constant sectional curvature -1.

4.2. ($\star$) The Bonnet – Myers theorem claims that if (M, g) is complete and connected, and there is $\varepsilon > 0$ such that $Ric_p(v) \geq \varepsilon$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.

Show by example that the assumption $\varepsilon > 0$ is essential (i.e. cannot be substituted by the assumption $Ric_p(v) > 0$).

Solution: One may consider an elliptic paraboloid of revolution $z = x^2 + y^2$. Its curvature is positive, but the paraboloid is not compact (e.g., it is unbounded). Note that although the curvature is positive (since the manifold is 2-dimensional sectional and Ricci curvatures coincide) it is not separated from zero, so there is no contradiction with Bonnet-Myers theorem.

4.3. Second Variational Formula of Energy

Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a proper variation of a geodesic $c : [a, b] \to M$, and let X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s, t)\|^2 dt.$$

Show that

$$E''(0) = \int_a^b \|\frac{D}{dt}X\|^2 - \langle R(X, c')c', X\rangle\, dt$$

Solution:

Since $E(s) = \frac{1}{2}\int_a^b \langle \frac{\partial F}{\partial t}(s, t), \frac{\partial F}{\partial t}(s, t)\rangle\, dt$, using the Riemannian property of covariant derivative we obtain

$$E'(s) = \int_a^b \left\langle \frac{D}{ds}\frac{\partial F}{\partial t}(s, t), \frac{\partial F}{\partial t}(s, t) \right\rangle dt.$$

Differentiating the integrand with respect to s, using the Symmetry Lemma, and setting then $s = 0$ yields

$$E''(0) = \int_a^b \frac{d}{ds}\Big|_{s=0} \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s, t), \frac{\partial F}{\partial s}(s, t) \right\rangle dt.$$

Applying Riemannian property of covariant derivative, Symmetry Lemma, and using that $\frac{\partial F}{\partial s}(0, t) = X(t)$, we conclude that

$$E''(0) = \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s, t), \frac{\partial F}{\partial t}(0, t) \right\rangle dt + \int_a^b \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(0, t), \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial t}(s, t) \right\rangle dt =$$
$$= \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s, t), \frac{\partial F}{\partial t}(0, t) \right\rangle dt + \int_a^b \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(0, t), \frac{D}{dt}\frac{\partial F}{\partial s}(0, t) \right\rangle dt =$$
$$= \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s, t), c'(t) \right\rangle dt + \int_a^b \left\| \frac{DX}{dt} \right\|^2 dt$$

Now we use Lemma 8.5 to interchange the order of covariant derivatives, and again Riemannian property to obtain

$$\left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle =$$

$$= \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle + \left\langle R\left(\frac{\partial F}{\partial s}(0,t), \frac{\partial F}{\partial t}(0,t)\right)\frac{\partial F}{\partial s}(0,t), c'(t) \right\rangle =$$

$$= \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle - \left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), \frac{D}{dt}c'(t) \right\rangle + \langle R(X(t), c'(t))X'(t), c'(t)\rangle =$$

$$= \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle - \langle R(X(t), c'(t))c'(t), X(t)\rangle,$$

since $c(t)$ is geodesic and $\frac{D}{dt}c'(t) = 0$.

Now we are left to show that

$$\int_a^b \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle dt = 0$$

Indeed,

$$\int_a^b \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle dt = \left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,b), c'(b) \right\rangle - \left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,a), c'(a) \right\rangle,$$

but

$$\frac{\partial F}{\partial s}(s,a) = \frac{\partial F}{\partial s}(s,b) = 0$$

since the variation $F(s,t)$ is proper.

4.4. Scalar curvature

The *scalar curvature* $s(p)$ at point $p \in M$ is defined by

$$s(p) = \sum_{j=1}^{n} Ric_p(u_j),$$

where $\{u_j\}$ is an orthonormal basis of $T_p(M)$.

(a) Let V be a vector space, $\langle \cdot, \cdot \rangle$ is an inner product on V, and Q is a quadratic form on V. Show that there exists a linear map $T \in \mathrm{End}(V)$ such that $Q(x) = \langle Tx, x\rangle$ for every $x \in V$.

(b) Show that the scalar curvature is well-defined, i.e. it does not depend on the choice of an orthonormal basis of $T_p(M)$.

Solution:

(a) Choose any orthonormal basis $\{e_i\}$ of V. Then $Q(x)$ can be written as $Q(x) = x^t G x$ for appropriate symmetric matrix $G = (g_{ij})$. Here $g_{ij} = \tilde{Q}(e_i, e_j)$, where $\tilde{Q}$ is the symmetric bilinear form constructed by Q, i.e. $\tilde{Q}(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$.
Since the basis is orthonormal, the inner product can be written as $\langle x, x\rangle = x^t x$. We need to find (a matrix) T such that $Q(x) = \langle Tx, x\rangle$, i.e. $x^t G x = (Tx)^t x$, or equivalently, $x^t G x = x^t T^t x$. This holds if $G = T^t$, or $T = G^t (= G$ since G is symmetric). It is easy to check that T is well-defined: if we change basis via an orthogonal transformation matrix P, then G in the new basis becomes PGP^t, and T becomes PTP^{-1}, which agree since $P^t = P^{-1}$ for orthogonal matrices.

(b) According to Lemma 7.9 from the lectures, Ric_p is a quadratic form. Thus, (a) implies that there exists $T \in \mathrm{End}(T_pM)$ such that $Ric_p(u) = \langle Tu, u\rangle$ for every $u \in T_pM$. Denote the matrix of T in the basis $\{u_j\}$ by (T_{ij}). Then

$$s(p) = \sum_{j=1}^{n} Ric_p(u_j) = \sum_{j=1}^{n} \langle Tu_j, u_j\rangle = \sum_{j=1}^{n}\langle \sum_{i=1}^{n} T_{ij}u_i, u_j\rangle = \sum_{j=1}^{n} T_{jj} = \mathrm{tr}\,(T)$$

which does not depend on the basis.