Durham University
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Riemannian Geometry IV, Homework 4 (Week 14)

Due date for starred problems: **Thursday, February 18**.

4.1. Constant sectional curvature of hyperbolic 3-space

Let $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ be the upper half-space model of the 3-dimensional hyperbolic space, i.e. its metric is defined by $g_{ij} = 0$ for $i \neq j$, $g_{ii} = 1/x_3^2$.

(a) Show that sectional curvatures $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$ and $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ in $\mathbb{H}^3$ are equal to -1.

(b) Use (a) and the linearity of the Riemann curvature tensor to show that for any $p \in \mathbb{H}^3$ and $v_1, v_2, v_3, v_4 \in T_p\mathbb{H}^3$

$$\langle R(v_1, v_2)v_3, v_4\rangle = -(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

holds.

(c) Use (b) to show that 3-dimensional hyperbolic space $\mathbb{H}^3$ has constant sectional curvature -1.

(d) Show that n-dimensional hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ with metric $g_{ij} = 0$ for $i \neq j$, $g_{ii} = 1/x_n^2$ has constant sectional curvature -1.

4.2. ($\star$) The Bonnet – Myers theorem claims that if (M, g) is complete and connected, and there is $\varepsilon > 0$ such that $Ric_p(v) \geq \varepsilon$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.

Show by example that the assumption $\varepsilon > 0$ is essential (i.e. cannot be substituted by the assumption $Ric_p(v) > 0$).

4.3. Second Variation Formula of Energy

Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a proper variation of a geodesic $c : [a, b] \to M$, and let X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s, t)\|^2 dt.$$

Show that

$$E''(0) = \int_a^b \|\frac{D}{dt}X\|^2 - \langle R(X, c')c', X\rangle dt$$

4.4. Scalar curvature

The *scalar curvature* $s(p)$ at point $p \in M$ is defined by

$$s(p) = \sum_{j=1}^{n} Ric_p(u_j),$$

where $\{u_j\}$ is an orthonormal basis of $T_p(M)$.

(a) Let V be a vector space, $\langle \cdot\,, \cdot\rangle$ is an inner product on V, and Q is a quadratic form on V. Show that there exists a linear map $T \in \mathrm{End}(V)$ such that $Q(x) = \langle Tx\,, x\rangle$ for every $x \in V$.

(b) Show that the scalar curvature is well-defined, i.e. it does not depend on the choice of an orthonormal basis of $T_p(M)$.