Durham University
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Riemannian Geometry IV, Solutions 5 (Week 15)

5.1. Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be a unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2, \pi/2] \to TS^2$ along c by

$$X(t) = (0, \cos t, 0).$$

Let $\frac{D}{dt}$ denote the covariant derivative along c.

(a) Calculate $\frac{D}{dt}X(t)$ and $\frac{D^2}{dt^2}X(t)$.
(b) Show that X satisfies the Jacobi equation.

Solution:

The problem can be solved by a direct computation: compute Christoffel symbols, and then compute first and second covariant derivatives of $X(t)$, then verify the Jacobi equation for $X(t)$.

(a) If we parametrize the sphere by $(x, y, z) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, one has $\Gamma^2_{11} = -\sin\vartheta\cos\vartheta$, $\Gamma^1_{12} = \Gamma^1_{21} = \cot\vartheta$ with others Γ^k_{ij} equal to zero, where $\varphi = x_1$ and $\vartheta = x_2$ (see Exercise 3.3).
In these coordinates, the curve $c(t) = (\cos t, 0, \sin t)$ is $c(t) = (0, \frac{\pi}{2} - t)$, $c'(t) = (0, -1) = -\frac{\partial}{\partial\vartheta}$. Further, observe that

$$\frac{\partial}{\partial\varphi}\Big|_{c(t)} = (-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, 0)\Big|_{\varphi=0,\,\vartheta=\frac{\pi}{2}-t} = (0, \cos t, 0) = X(t)$$

Hence,

$$\frac{D}{dt}X(t) = \nabla_{c'(t)}X(t) = \nabla_{-\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\varphi} = -\cot\vartheta\frac{\partial}{\partial\varphi}\Big|_{c(t)} = -\tan t\, X(t),$$

$$\frac{D^2}{dt^2}X(t) = \frac{D}{dt}(-\tan t\, X(t)) = -\sec^2 t\, X(t) + \tan^2 t\, X(t) = -X(t) = -\frac{\partial}{\partial\varphi}\Big|_{c(t)}$$

(b) Compute $R(X, c')c' = \nabla_X\nabla_{c'}c' - \nabla_{c'}\nabla_X c' - \nabla_{[X,c']}c'$. As $X = \frac{\partial}{\partial\varphi}$ and $c' = -\frac{\partial}{\partial\vartheta}$, we have $[X, c'] = 0$. Also,

$$\nabla_X\nabla_{c'}c' = \nabla_{\frac{\partial}{\partial\varphi}}\nabla_{-\frac{\partial}{\partial\vartheta}} - \frac{\partial}{\partial\vartheta} = \nabla_{\frac{\partial}{\partial\varphi}}0 = 0,$$

$$\nabla_{c'}\nabla_X c' = \nabla_{-\frac{\partial}{\partial\vartheta}}\nabla_{\frac{\partial}{\partial\varphi}} - \frac{\partial}{\partial\vartheta} = \nabla_{\frac{\partial}{\partial\vartheta}}(\cot\vartheta\frac{\partial}{\partial\varphi}) = -\frac{1}{\sin^2\vartheta}\frac{\partial}{\partial\varphi} + \cot\vartheta(\cot\vartheta\frac{\partial}{\partial\varphi}) = (\cot^2\vartheta - \frac{1}{\sin^2\vartheta})\frac{\partial}{\partial\varphi} = -X(t).$$

Thus, $R(X, c')c' = X(t) = \frac{\partial}{\partial\varphi}$, and (since $\frac{D^2}{dt^2}X(t) = -X(t) = -\frac{\partial}{\partial\varphi}$) Jacobi equation holds.

5.2. ($\star$) Choose any $r > 0$ and consider a cylinder $C \subset \mathbb{R}^3$ with induced metric,

$$C = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = r^2\}$$

C can be parametrized by

$$(r\cos\varphi, r\sin\varphi, z), \quad \varphi \in [0, 2\pi), z \in \mathbb{R}$$

(a) Show that a curve $c(t) = (r\cos(t/r), r\sin(t/r), 0)$ is a geodesic. Write $c(t)$ in the form $(\varphi(t), z(t))$.
(b) Let $\alpha \in \mathbb{R}$. Show that $c_\alpha(t) = (\varphi(t), z(t)) = ((t\cos\alpha)/r, t\sin\alpha)$ is a geodesic.
(c) Construct two distinct geodesic variations $F_1(s, t)$ and $F_2(s, t)$ of $c(t)$, such that $F_1(s, 0) \equiv c(0)$, and $F_2(s, 0) \neq c(0)$ for any $s \neq 0$. Compute the variational vector fields of F_1 and F_2.
(d) Construct the basis of the space J_c of Jacobi fields along $c(t)$.
(e) Show that for any $t_0 \in \mathbb{R}$ the points $c(0)$ and $c(t_0)$ are not conjugate along $c(t)$.

Solution:

(a) One way to do this is to use symmetry of C. More precisely, the reflection in the plane $z = 0$ is obviously an isometry of C, and it preserves $c(t)$. By the uniqueness theorem of a geodesic in a given direction, the trace of $c(t)$ should be a trace of a geodesic. Now observe that $\|c'(t)\| = 1$, so $c(t)$ is a geodesic.

Another way is to observe that C is locally isometric to $\mathbb{R}^2$, and the isometry takes $c(t)$ to a straight line on $\mathbb{R}^2$.

Finally, one can compute the induced metric and Christoffel symbols (they are all zeros!), and then verify that $c(t)$ satisfies the ODE for geodesics.

In coordinates (φ, z), the geodesic $c(t)$ is written as $c(t) = (t/r, 0)$.

(b) The second and the third methods from (a) work perfectly fine in this case as well.

(c) We can take

$$F_1(s,t) = \left(r\cos\left(\frac{t\cos s}{r}\right), r\sin\left(\frac{t\cos s}{r}\right), t\sin s\right)$$

Clearly, $F_1(0,t) = c(t)$, $F_1(s,0) \equiv (r,0,0) = c(0)$, and every $t \mapsto F_1(s_0,t)$ is a geodesic by (b). The variational vector field is $X_1(t) = (0,0,t)$.

Shifting $c(t)$ in vertical direction, we can take

$$F_2(s,t) = (r\cos(t/r), r\sin(t/r), s)$$

The corresponding variational vector field is $X_2(t) = (0,0,1)$.

(d) We need $2n = 4$ linearly independent vector fields. We have already found two, and observe that X_1 and X_2 are both orthogonal and clearly linear independent, so they form a basis of the space of orthogonal Jacobi fields. We can also take $X_3(t) = c'(t)$ and $X_4 = tc'(t)$, all of them together form a basis.

(e) Assume that $J(0) = J(t_0) = 0$ for some $J \in J_c$. Since $J(0) = 0$, J should be a linear combination of X_1 and X_3. However, such a non-zero linear combination never vanishes except for $t = 0$.

5.3. Jacobi fields on manifolds of constant curvature.

Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic parametrized by arc length. Let $J : [0,1] \to TM$ be an orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t)\rangle = 0$ for every $t \in [0,1]$).

(a) Show that $R(J,c')c' = KJ$.

(b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \frac{DJ}{dt}(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}}Z_2(t) & \text{if } K < 0. \end{cases}$$

Hint: Show that these fields satisfy Jacobi equation, there value and covariant derivative at $t = 0$ is the same as for $J(t)$.

Solution:

(a) We conclude from Exercise 3.4 that

$$R(v_1,v_2)v_3 = K(\langle v_2,v_3\rangle v_1 - \langle v_1,v_3\rangle v_2).$$

This implies

$$R(J,c')c' = K(\langle c',c'\rangle J - \langle J,c'\rangle c').$$

Since $\|c'\|^2 = 1$ and $J \perp c'$, we obtain

$$R(J,c')c' = KJ.$$

(b) We only consider the case $K > 0$, all other cases are similar. The vector field

$$J(t) = \cos(t\sqrt{K}) Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t)$$

satisfies $J(0) = Z_1(0)$ and

$$\frac{DJ}{dt}(t) = -\sqrt{K}\sin(t\sqrt{K}) Z_1(t) + \cos(t\sqrt{K}) Z_2(t),$$

which implies $\frac{DJ}{dt}(0) = Z_2(0)$. Obviously, we have

$$\frac{D^2J}{dt^2}(t) = -K\cos(t\sqrt{K}) Z_1(t) - \sqrt{K}\sin(t\sqrt{K}) Z_2(t) = -KJ(t),$$

and therefore we obtain

$$\frac{D^2J}{dt^2}(t) + KJ(t) = 0,$$

i.e., J satisfies the Jacobi equation.