Durham University
Pavel Tumarkin

Epiphany 2016

Riemannian Geometry IV, Homework 5 (Week 15)

Due date for starred problems: **Thursday, March 3**.

5.1. Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be a unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2, \pi/2] \to TS^2$ along c by

$$X(t) = (0, \cos t, 0).$$

Let $\frac{D}{dt}$ denote the covariant derivative along c.

(a) Calculate $\frac{D}{dt}X(t)$ and $\frac{D^2}{dt^2}X(t)$.

(b) Show that X satisfies the Jacobi equation.

5.2. ($\star$) Choose any $r > 0$ and consider a cylinder $C \subset \mathbb{R}^3$ with induced metric,

$$C = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = r^2\}$$

C can be parametrized by

$$(r\cos\varphi, r\sin\varphi, z), \quad \varphi \in [0, 2\pi), z \in \mathbb{R}$$

(a) Show that a curve $c(t) = (r\cos(t/r), r\sin(t/r), 0)$ is a geodesic. Write $c(t)$ in the form $(\varphi(t), z(t))$.

(b) Let $\alpha \in \mathbb{R}$. Show that $c_\alpha(t) = (\varphi(t), z(t)) = ((t\cos\alpha)/r, t\sin\alpha)$ is a geodesic.

(c) Construct two distinct geodesic variations $F_1(s,t)$ and $F_2(s,t)$ of $c(t)$, such that $F_1(s,0) \equiv c(0)$, and $F_2(s,0) \neq c(0)$ for any $s \neq 0$. Compute the variational vector fields of F_1 and F_2.

(d) Construct the basis of the space J_c of Jacobi fields along $c(t)$.

(e) Show that for any $t_0 \in \mathbb{R}$ the points $c(0)$ and $c(t_0)$ are not conjugate along $c(t)$.

5.3. Jacobi fields on manifolds of constant curvature.
Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic parametrized by arc length. Let $J : [0,1] \to TM$ be an orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t)\rangle = 0$ for every $t \in [0,1]$).

(a) Show that $R(J, c')c' = KJ$.

(b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \frac{DJ}{dt}(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}}Z_2(t) & \text{if } K < 0. \end{cases}$$

Hint: Show that these fields satisfy Jacobi equation, and there value and covariant derivative at $t = 0$ is the same as for $J(t)$.