Durham University
Pavel Tumarkin

Epiphany 2016

Riemannian Geometry IV, Solutions 7 (Week 17)

7.1. ($\star$) Let M be a Riemannian manifold of non-positive sectional curvature, i.e. $K(\Pi) \le 0$ for any 2-plane $\Pi \subset TM$.

(a) Let $c : [a, b] \to M$ be a geodesic and let J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \ge 0$, i.e., f is a convex function.

(b) Derive from (a) that M does not admit conjugate points.

Solution:

(a) We have

$$f'(t) = \frac{d}{dt}\Big|_{t=0} \langle J(t), J(t)\rangle = 2\langle \frac{D}{dt} J(t), J(t)\rangle$$

and

$$f''(t) = 2\left(\langle \frac{D^2}{dt^2} J(t), J(t)\rangle + \left\|\frac{D}{dt} J(t)\right\|^2\right).$$

Using Jacobi equation, we conclude

$$f''(t) = 2\left(-\langle R(J(t), c'(t))c'(t), J(t)\rangle + \left\|\frac{D}{dt} J(t)\right\|^2\right).$$

We have $\langle R(J(t), c'(t))c'(t), J(t)\rangle = 0$ if $J(t), c'(t)$ are linear dependent and, otherwise, for $\Pi = \mathrm{span}(J(t), c'(t)) \subset T_{c(t)}M$,

$$\langle R(J(t), c'(t))c'(t), J(t)\rangle = K(\Pi)\left(\|J(t)\|^2\|c'(t)\|^2 - (\langle J(t), c'(t)\rangle)^2\right) \le 0,$$

since sectional curvature is non-positive. This shows that $f''(t)$, as a sum of two non-negative terms, is greater than or equal to zero.

(b) If there were a conjugate point $q = c(t_2)$ to a point $p = c(t_1)$ along the geodesic c, then we would have a non-vanishing Jacobi field J along c with $J(t_1) = 0$ and $J(t_2) = 0$. This would imply that the convex, non-negative function $f(t) = \|J(t)\|^2$ would have zeros at $t = t_1$ and $= t_2$. This would force f to vanish identically on the interval $[t_1, t_2]$, which would imply that J vanishes as well, which leads to a contradiction.

7.2. ($\star$) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ be a paraboloid of revolution with metric induced by $\mathbb{R}^3$. Let $p = (0, 0, 0)$. Show that p has no conjugate points in M.

Solution:

Let $q = (q_1, q_2, q_3) \ne p$ be any point in M. Denote by $\Pi \in \mathbb{R}^3$ the 2-dimensional plane spanned by q and the z-axis. It is easy to check that there is a geodesic $c(t) \in M \cap \Pi$ with $c(0) = p$, $c(t_1) = q$. Moreover, the argument used in class (vertical geodesics in $\mathbb{H}^2$) shows that $c(t)$ is a minimal geodesic between p and q. By Theorem 9.24 this implies that for any $t_0 \in (0, t_1)$ the point $c(t_0)$ is not conjugate to p.

Rotating the whole picture around the z-axis (this is clearly an isometry of M) we see that p has no conjugate points in a ball $z < q_3$, so taking q far enough from p we can prove that p has no conjugate points in a ball of any size centered at p.

7.3. Let (M, g) be a Riemannian manifold. For a tensor T let ∇T denote its covariant derivative, see Exercise 9.3. T is called a *parallel tensor* if $\nabla T = 0$.

(a) Assume that $T_1, T_2 : \mathfrak{X} \times \mathfrak{X} \to C^\infty(M)$ are parallel tensors. Show that the tensor $T : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2)T_2(X_3, X_4),$$

is also parallel.

(b) Use (a) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle.$$

(c) Use Exercise 3.4 and (b) to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle.$$

Solution:

(a) We have

$$\begin{aligned}
\nabla T(X_1, X_2, X_3, X_4, Y) = & \\
&= Y(T_1(X_1, X_2)T_2(X_3, X_4)) - \sum_{i=1}^{4} T(X_1, \dots, \nabla_Y X_i, \dots, X_4) = \\
&= T_1(X_1, X_2)\underbrace{(Y(T_2(X_3, X_4)) - T_2(\nabla_Y X_3) - T_2(\nabla_Y X_4))}_{=\nabla T_2(X_3,X_4,Y)=0} + \\
&\quad + T_2(X_3, X_4)\underbrace{(Y(T_1(X_1, X_2)) - T_1(\nabla_Y X_1) - T_1(\nabla_Y X_2))}_{=\nabla T_1(X_1,X_2,Y)=0} = 0.
\end{aligned}$$

(b) Let $T(X, Y) = \langle X, Y\rangle$. Since ∇ is Riemannian, we have

$$\nabla T(X, Y, Z) = Z(\langle X, Y\rangle) - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0.$$

Note that $R'(X, Y, Z, W) = T(X, W)T(Y, Z) - T(X, Z)T(Y, W)$. Part (a) implies then that we have $\nabla R' = 0$.

(c) If (M, g) is a manifold with constant sectional curvature $K_0 \in \mathbb{R}$, we have by Exercise 3.4

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle = K_0(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle) = K_0 R'(X, Y, Z, W).$$

Then $\nabla R = K_0 \nabla R' = 0$ follows from (b).