

Riemannian Geometry IV, Homework 7 (Week 17)

Due date for starred problems: **Thursday, March 17.**

7.1. (★) Let M be a Riemannian manifold of non-positive sectional curvature, i.e. $K(\Pi) \leq 0$ for any 2-plane $\Pi \subset TM$.

- (a) Let $c : [a, b] \rightarrow M$ be a geodesic and let J be a Jacobi field along c . Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \geq 0$, i.e., f is a convex function.
- (b) Derive from (a) that M does not admit conjugate points.

7.2. (★) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ be a paraboloid of revolution with metric induced by \mathbb{R}^3 . Let $p = (0, 0, 0)$. Show that p has no conjugate points in M .

7.3. Let (M, g) be a Riemannian manifold. For a tensor T let ∇T denote its covariant derivative, see Exercise 9.3. T is called a *parallel tensor* if $\nabla T = 0$.

- (a) Assume that $T_1, T_2 : \mathfrak{X} \times \mathfrak{X} \rightarrow C^\infty(M)$ are parallel tensors. Show that the tensor $T : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \rightarrow C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2)T_2(X_3, X_4),$$

is also parallel.

- (b) Use (a) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle.$$

- (c) Use Exercise 3.4 and (b) to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle.$$