Durham University
Pavel Tumarkin

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Riemannian Geometry IV, Term 2 (Section 10, non-examinable)

10 Curvature and geometry

10.1 Index form

Definition 10.1. Recall (see the proof of Second Variational Formula) that given a geodesic $c : [0, a] \to M$ there exists a symmetric bilinear form on $\mathfrak{X}_c(M)$ given by $I_a(V, W) = \int\limits_0^a (\langle V, W\rangle + \langle R(V, c')c', W\rangle)dt$.

The quadratic form $I_a(V, V)$ is called an index form.

Definition 10.2. The index of I_a is the maximal dimension of a subspace of $\mathfrak{X}_c(M)$ on which I_a is negative definite (i.e., negative inertia index).

Theorem 10.3 (Morse Index Theorem)**.** *The index of I_a is finite and equals the number of points $c(t)$, $0 < t < a$, conjugate to $c(0)$ counted with multiplicities.*

Corollary 10.4. *The set of conjugate points along a geodesic is discrete.*

Lemma 10.5 (Index Lemma)**.** *Let $c : [0, a] \to M$ be a geodesic containing no conjugate points to $c(0)$. Let $J \in J_c$ be an orthogonal Jacobi field. Let V be a piecewise smooth vector field on c, $\langle V, c'\rangle = 0$. Suppose also $J(0) = V(0) = 0$, $J(t_0) = V(t_0)$ for some $t_0 \in (0, a]$.*

Then $I_{t_0}(J, J) \le I_{t_0}(V, V)$, where equality holds only if $V = J$ on $[0, a]$.

10.2 Rauch Comparison Theorem

Theorem 10.6 (Rauch Comparison Theorem)**.** *Let $c : [0, a] \to M^n$ and $\tilde{c} : [0, a] \to \widetilde{M}^m$ be two unit speed geodesics, and let $J : [0, a] \to TM$ and $\widetilde{J} : [0, a] \to T\widetilde{M}$ be two orthogonal Jacobi fields along c and $\widetilde{c}$ with $J(0) = \widetilde{J}(0) = 0$, $\|\frac{D}{dt}J(0)\| = \|\frac{D}{dt}\widetilde{J}(0)\|$. Assume that $\widetilde{J}$ does not have conjugate points on $(0, a)$, and that for any $t \in [0, a]$ the inequality $K_M(\Pi) \le K_{\widetilde{M}}(\widetilde{\Pi})$ holds for all 2-planes $\Pi \subset T_{c(t)}M$ and $\widetilde{\Pi} \subset T_{\tilde{c}(t)}\widetilde{M}$. Then $\|J(t)\| \ge \|\widetilde{J}(t)\|$ for all $t \in [0, a]$.*

Corollary 10.7. *Let M satisfy $0 < K_{\min} \le K \le K_{\max}$, $c : [0, a] \to M$ is a geodesic. Then for any two conjugate points along c the distance d between them satisfies*

$$\frac{\pi}{\sqrt{K_{\max}}} \le d \le \frac{\pi}{\sqrt{K_{\min}}}$$

10.3 Injectivity radius

Definition 10.8. The injectivity radius of a point $p \in M$ is $i_p = \sup\{r \ge 0 \mid \exp_p \text{is diffeo in } B_r(0_p)\} = \inf\limits_{q \in C_m(p)} d(p, q)$, where $C_m(p)$ is the cut locus of p.

The injectivity radius of M is $i(M) = \inf\limits_p i_p = \inf\limits_{p \in M} d(p, C_m(p))$.

Example 10.9. $i(S^2) = \pi$; $i(\mathbb{R}^2) = i(\mathbb{H}^2) = \infty$; $i(\mathbb{T}^2) = 1/2$; $i(M) = 0$ for any non-complete M.

Proposition 10.10. *Let M be complete with sectional curvature K satisfying $0 < K_{\min} \le K \le K_{\max}$. Then at least one of the following holds.*

(a) $i(M) \geq \pi/\sqrt{K_{max}}$*, or*

(b) there exists a shortest closed geodesic $c \subset M$ *s.t.* $i(M) = \frac{1}{2}l(c)$*.*

Lemma 10.11 (Klingenberg, 1961)**.** *Let* M *be a compact simply-connected Riemannian manifold of dimension* $n \geq 3$*, and let* $1/4 < K \leq 1$*. Then* $i(M) \geq \pi$*.*

Remark. If n is even and M is orientable then it suffices for M to satisfy $0 < K \leq 1$.

10.4 Sphere Theorem

Theorem 10.12 (Berger, Klingenberg, 1961)**.** *Let* M *be a compact simply-connected Riemannian* n*-dimensional manifold with* $\frac{1}{4} < K \leq 1$*. Then* M *is* <u>*homeomorphic*</u> *to* S^n*.*

Remark. (a) In fact, a stronger result is valid: M is diffeomorphic to S^n (Brendle, Schoen, 2009).

(b) The Sphere Theorem does not hold in assumptions $\frac{1}{4} \leq K(\Pi) \leq 1$.

(c) The theorem obviously holds in assumptions $\frac{\delta}{4} < K(\Pi) \leq \delta$ for any $\delta > 0$.

(d) In dimension $n = 2$ stronger result holds: if $K \geq 0$ for all $p \in M$ and $K > 0$ in at least one point, then M is homeomorphic to S^2.

The proof of the Sphere Theorem is based on the following two lemmas.

Lemma 10.13. *Let* M *be a compact Riemannian manifold, let* $p, q \in M$ *be such that* diam $M = d(p,q)$*. Then for any* $w \in T_M$ *there exists a minimal geodesic* $c : [0, d(p,q)] \to M$*,* $c(0) = p$*,* $c(d(p,q)) = q$*, such that* $\langle w, c'(0)\rangle \geq 0$*.*

Lemma 10.14. *Let* M *be a compact simply-connected Riemannian manifold with sectional curvature satisfying* $\frac{1}{4} < \delta \leq K \leq 1$*, let* $p, q \in M$ *be such that* diam $M = d(p,q)$*. Choose any* $\rho \in (\pi/2\sqrt{\delta}, \pi)$*. Then* $M = B_\rho(p) \cup B_\rho(q)$*.*

10.5 Spaces of constant curvature

Theorem 10.15. *Let* M *be a complete simply-connected Riemannian manifold of* <u>*constant*</u> *sectional curvature* K*. Then*
1) if $K > 0$ *then* M *is isometric to* S^n *(assuming* $K = 1$*);*
2) if $K = 0$ *then* M *is isometric to* $\mathbb{E}^n$*;*
3) if $K < 0$ *then* M *is isometric to* $\mathbb{H}^n$ *(assuming* $K = -1$*).*

10.6 Comparison triangles

Definition 10.16. A <u>triangle</u> in a Riemannian manifold is a collection of 3 points with <u>minimal</u> geodesics connecting them. A <u>generalized triangle</u> is a collection of 3 points with <u>any</u> geodesics connecting them and satisfying triangle inequality.

Definition 10.17. A <u>comparison triangle</u> $p'q'r'$ for a generalized triangle $pqr \in M$ is a triangle in a space of constant curvature with sides of the same lengths.

Theorem 10.18 (Alexandrov, Toponogov, 1959)**.** *Let* $K(\Pi) \geq 0$ *for all* $\Pi \in T_pM$ *for all* $p \in M$*. Let* $p_0, p_1, p_2 \in M$*. Let* p_3 *lie between* p_1 *and* p_2 *(i.e.* $d(p_1,p_3) + d(p_2,p_3) = d(p_1,p_2)$*). Let* p'_0, p'_1, p'_2 *be a comparison triangle in* $\mathbb{E}^2$*. Define* p'_3 *by* $d(p_i,p_3)_M = d(p'_i,p'_3)_{\mathbb{E}^2}$ *for* $i = 1,2$*. Then* $d(p_0,p_3)_M \geq d(p'_0,p'_3)_{\mathbb{E}^2}$ *(Alexandrov – Toponogov inequality). Conversely, if Alexandrov – Toponogov inequality holds for all* p_0, p_1, p_2, p_3 *then* $K \geq 0$*.*

Remark. (a) Dual statement for $K \leq 0$ with inverse AT-inequality.

(b) Equivalent conditions:

- smaller K implies smaller <u>angles</u>;
- smaller K implies bigger <u>circumference</u> of a circle of radius r;
- smaller K implies bigger <u>volume</u> of a ball or radius r.