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Riemannian Geometry IV, Term 2 (Section 6)

6 Crash course: Basics about Lie groups

6.1 Left-invariant vector fields and Lie algebra

Definition 6.1. A Lie group G is a smooth manifold with a smooth group structure, i.e. the maps $G \times G \to G$, $(g,h) \mapsto gh$ and $G \to G$, $g \mapsto g^{-1}$ are smooth.

Examples. Matrix Lie groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$.

Definition 6.2. Let G a be a Lie group, $g \in G$. Then the maps $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ are called left- and right-translation. L_g and R_g are diffeomorphisms of G.

Remark. (a) $L_{g^{-1}} \circ L_g = id_G$, $\quad L_{g_1} R_{g_2}(h) = R_{g_2} L_{g_1}(h) = g_1 h g_2$.

(b) The differential $DL_g : T_h G \to T_{gh} G$ gives a natural identification of tangent spaces.

Example 6.3. Let $G \subset GL_n(\mathbb{R})$ be a matrix group, $v \in T_e G$. Then $DL_g(e)v = gv$.

Definition 6.4. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if for any $g \in G$ $\ DL_g X = X \circ L_g$, i.e. $DL_g(h)X(h) = X(gh)$.

Remark 6.5. (a) Left-invariant vector fields on G form a vector space over $\mathbb{R}$.

(b) Left-invariant vector field is determined by its value at e: $X(g) = DL_g(e)X(e)$.

(c) Hence, the space of left-invariant vector fields on G can be identified with $T_e G$.

Definition 6.6. The space of left-invariant vector fields on G is called the Lie algebra of G and denoted by $\mathfrak{g}$.

Lemma 6.7. *Let M, N be smooth manifolds, $X \in \mathfrak{X}(M)$, $f \in C^\infty(N)$, $p \in M$, and let $\varphi : M \to N$ be a smooth map. Then*

$$(d\varphi(p)X(p))f = X(p)(f \circ \varphi)$$

Proposition 6.8. *Let X be a Lie group with Lie algebra $\mathfrak{g}$. Then for any $X, Y \in \mathfrak{g}$ the Lie bracket $[X,Y] \in \mathfrak{g}$. Consequently, $\mathfrak{g}$ is indeed a Lie algebra (see Definition 2.22).*

6.2 Lie group exponential map and adjoint representation

Definition 6.9. Define $\mathrm{Exp} : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $\mathrm{Exp}\,(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

Properties. (a) The infinite sum converges for any matrix $A \in M_n(\mathbb{R})$, so $\mathrm{Exp}\,(A)$ is well-defined;

(b) $\mathrm{Exp}\,(0) = I$;

(c) if $AB = BA$ then $\mathrm{Exp}\,(A+B) = \mathrm{Exp}\,(A) \cdot \mathrm{Exp}\,(B)$; in particular, $\mathrm{Exp}\,(-A)\mathrm{Exp}\,(A) = I$, so $\mathrm{Exp}\,(A) \in GL_n(\mathbb{R})$ for any $A \in M_n(\mathbb{R})$.

Example 6.10. Computation of the exponent for a diagonalizable matrix.

Proposition 6.11. *Let G be a matrix Lie group. Let $v \in T_eG$ and let X be the unique left-invariant vector field on G with $X(e) = v$. Then the curve $c(t) = \mathrm{Exp}\,(tv) \in G$ satisfies $c(0) = e$, $c'(0) = v$ and $c'(t) = X(c(t))$.*

A curve of the form $c(t) = \mathrm{Exp}\,(tv)$ is called a 1-parameter subgroup of G with $c'(0) = v$.

Remark. For an abstract Lie group the exponential map can be defined as follows. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $v \in T_eG$ and let $X \in \mathfrak{g}$ be the unique left-invariant vector field with $X(e) = v$. Then there exists a unique curve $c_v : \mathbb{R} \to G$ with $c_v(0) = e$, $c_v'(t) = X(c_v(t))$ [without proof]. The curve c_v is called an integral curve of X. We define the exponential map by $\mathrm{Exp}\,(v) = c_v(1)$.

Definition 6.12. Let G be a Lie group. For $g \in G$ the adjoint representation $\mathrm{Ad}_g : T_eG \to T_eG$ is defined by

$$\mathrm{Ad}_g(w) = \frac{d}{dt}\Big|_{t=0} L_g R_{g^{-1}}(\mathrm{Exp}\,(tw)) = \frac{d}{dt}\Big|_{t=0} g\mathrm{Exp}\,(tw)g^{-1}.$$

For $v \in T_eG$ the adjoint representation $\mathrm{ad}_v : T_eG \to T_eG$ is defined by

$$\mathrm{ad}_v(w) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\mathrm{Exp}\,(tv)}(w) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \mathrm{Exp}\,(tv)\mathrm{Exp}\,(sw)\mathrm{Exp}\,(-tv).$$

Theorem 6.13 (without proof)**.** *Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Then for all $X, Y \in \mathfrak{g}$ holds $\mathrm{ad}_{X(e)}Y(e) = [X,Y](e) \in T_eG$, i.e. by canonical identification of $\mathfrak{g}$ with T_eG we have $\mathrm{ad}_X Y = [X,Y]$.*

Example 6.14. Theorem 6.13 for the case of a matrix Lie group.

6.3 Riemannian metrics on Lie groups

Definition 6.15. For a given inner product $\langle\cdot,\cdot\rangle_e$ on T_eG, define the inner product at $g \in G$ for $v, w \in T_gG$ by $\langle v, w\rangle_g = \langle DL_{g^{-1}}(g)v, DL_{g^{-1}}(g)w\rangle_e$. The family $(\langle\cdot,\cdot\rangle_g)_{g\in G}$ of inner products defines a left-invariant Riemannian metric on G.

Remark 6.16. Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with a left-invariant metric. Then

(a) the diffeomorphisms $L_g : G \to G$ are isometries;

(b) for any two left-invariant vector fields $X, Y \in \mathfrak{g}$ the function $g \mapsto \langle X(g), Y(g)\rangle_g$ is constant.

Theorem 6.17 (without proof)**.** *Let G be a compact Lie group. Then G admits a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle_g$, i.e. both families of diffeomorphisms L_g and R_g are isometries.*

Corollary 6.18. *Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with bi-invariant metric, let $X, Y, Z \in \mathfrak{g}$. Then $\langle [X,Y], Z\rangle = -\langle [X,Z], Y\rangle$.*

Corollary 6.19. *Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with bi-invariant metric and let ∇ be the Levi-Civita connection. Then for $X, Y \in \mathfrak{g}$ holds $\nabla_X Y = \frac{1}{2}[X,Y]$.*

Corollary 6.20. *(a) 1-parameter subgroups are exactly the geodesics of the bi-invariant metric on G;*

(b) the Lie group exponential map Exp coincides with the Riemannian exponential map $\exp_e$ at the identity.