

Riemannian Geometry IV, Term 2 (Sections 7-8)

7 Curvature

7.1 Riemann curvature tensor

Definition 7.1. Let (M, g) be a Riemannian manifold, let $\mathfrak{X}(M)$ be the space of vector fields on M , and let ∇ be the Levi-Civita connection. Define a map (Riemann curvature tensor) $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.

Remark. R is linear in all variables, so, it is a tensor; moreover, $R(fX, gY)hZ = fghR(X, Y)Z$ for any $f, g, h \in C^\infty(M)$.

Lemma 7.2. R has the following symmetries:

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| <p>(a) $R(X, Y)Z = -R(Y, X)Z$</p> <p>(b) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
(first Bianchi Identity)</p> | <p>(c) $\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$</p> <p>(d) $\langle R(X, Y)Z, W \rangle = -\langle R(Z, W)X, Y \rangle$</p> |
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Definition 7.3. Define components of Riemann curvature tensor $R_{ijkl} = \langle R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \rangle$, and R^l_{ijk} by $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_l R^l_{ijk} \frac{\partial}{\partial x_l}$.

$$\text{Then } R_{ijkl} = \sum_m R^l_{ijk} g_{ml} \quad \text{and} \quad R^l_{ijk} = \sum_m R_{ijkm} g^{ml}.$$

Example 7.4. Computation of components R_{ijks} and R^l_{ijk} for hyperbolic plane (in the upper half-plane model).

7.2 Sectional curvature

Definition 7.5. Let (M, g) be a Riemannian manifold, $p \in M$, $v_1, v_2 \in T_p M$, and let $\Pi \subset T_p M$ be the 2-plane spanned by v_1, v_2 .

The sectional curvature of Π at p is $K(\Pi) = K(v_1, v_2) = \frac{\langle R(v_1, v_2)v_2, v_1 \rangle}{\|v_1\|^2 \|v_2\|^2 - \langle v_1, v_2 \rangle^2}$.

Proposition 7.6. $K(\Pi)$ does not depend on the basis $\{v_1, v_2\}$ of Π .

Examples. Sectional curvature of a 2-sphere and hyperbolic plane.

7.3 Ricci curvature

Given $v, w \in T_p M$ define a linear map $R(\cdot, v)w : T_p M \rightarrow T_p M$ by $u \mapsto R(u, v)w$.

Definition 7.7. Ricci curvature tensor $Ric(v, w)$ is the trace of the map $R(\cdot, v)w$: $Ric_p(v, w) = tr(R(\cdot, v)w)$.

In an orthonormal basis $\{u_i\}$, $Ric_p(v, w) = \sum_{j=1}^n \langle R(u_j, v)w, u_j \rangle$.

Definition 7.8. Ricci curvature at p is $Ric_p(v) = Ric_p(v, v) = \sum_{j=1}^n \langle R(u_j, v)w, u_j \rangle$

In an orthonormal basis $\{v = u_1, \dots, u_n\}$ we have $Ric_p(v) = \sum_{j=2}^n K(v, u_j)$.

Lemma 7.9. $Ric(v, u)$ is a symmetric bilinear form (i.e. $Ric(v)$ is a quadratic form).

Example. If $K(v, w)$ is constant ($= K$) and $\|v\| = 1$, then $Ric(v) = (n - 1)K$.

8 Bonnet – Myers Theorem

Theorem 8.1 (Bonnet – Myers, 1935). *Let (M, g) be a connected, complete Riemannian manifold of dimension n .*

Suppose that $\text{Ric}(v) \geq \frac{n-1}{r^2}$ for all $v \in SM = \{w \in TM \mid \|w\| = 1\}$. Then $\text{diam } M (= \sup_{p,q \in M} d(p, q)) \leq \pi r$.

In particular, M is bounded, so, it is compact (as it is complete).

Theorem 8.2 (Second variation formula of length). *Let $c : [a, b] \rightarrow M$ be a geodesic parametrized by arc length, let $F : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$ be a proper variation of c , let $X(t) = \frac{\partial F}{\partial s}(0, t)$ be the variational vector field. Define $X^\perp(t) = X(t) - \langle X(t), c'(t) \rangle c'(t)$, the orthogonal component of $X(t)$. Let $l(s)$ be the length of the variation.*

$$\text{Then } l''(0) = \int_a^b (\| \frac{DX^\perp}{dt} \|^2 - K(c', X^\perp) \|X^\perp\|^2) dt.$$

Remark. In the case if X is collinear to c' (i.e. $X^\perp = 0$) we define $K(c', X^\perp) = 0$.

Corollary 8.3. *If $K(\Pi) < 0$ for every $p \in M$ and every 2-plane $\Pi \subset T_p M$ then every geodesic is locally minimal.*

Example 8.4. For the n -dimensional sphere S_r^n of radius r the inequality in the Bonnet – Myers Theorem becomes an equality. Hence, the bound is sharp.

Lemma 8.5. *Let $F(s, t)$ be a variation of a geodesic $c(t)$, and let $Z(s, t) \in T_{F(s,t)} M$ be smooth. Then $\frac{D}{ds} \frac{D}{dt} Z - \frac{D}{dt} \frac{D}{ds} Z = R(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}) Z$.*

Example 8.6. Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ be an n -dimensional torus with arbitrary metric g (compatible with the smooth structure). Then there exists $p \in T^n$ and $v \in T_p T^n$ such that $\text{Ric}_p(v) \leq 0$.