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Riemannian Geometry IV, Term 2 (Section 9)

9 Jacobi fields

9.1 Jacobi fields and geodesic variations

Definition 9.1. Let $c(t)$ be a geodesic. A vector field $J \in \mathfrak{X}_c(M)$ is a Jacobi field if it satisfies Jacobi equation: $\frac{D^2}{dt^2}J + R(J, c')c' = 0$.

Example 9.2. Vector fields $c'(t)$ and $tc'(t)$ are Jacobi fields for any geodesic $c(t)$.

Theorem 9.3. *Let $c(t)$ be a geodesic. Let $F(s,t)$ be a variation, s.t. every curve $F_s(t)$ is geodesic. Then the variational vector field $X(t) = \frac{\partial F}{\partial s}(0,t)$ is a Jacobi field.*

Example 9.4. Geodesic variation on a sphere and its variational vector field.

Definition 9.5. Let $E_1(t), \dots, E_n(t) \in \mathfrak{X}_c(M)$ be vector fields along $c(t)$. We say that $\{E_, \dots, E_n\}$ is a parallel orthonormal basis along c if for all t, i, j holds $\frac{D}{dt}E_i = 0$ and $\langle E_i, E_j \rangle = \delta_{ij}$.

Notation. $R_{ij} = \langle R(E_i, c')c', E_j \rangle$, R_{ij} is an $n \times n$ symmetric matrix depending on t.

Theorem 9.6. *Let $c(t)$ be a geodesic and $\{E_i\}$ be a parallel orthonormal basis along c. Take $J \in \mathfrak{X}_c(M)$ and its expansion $J = \sum_j J_j(t)E_j(t)$ (where $J_j(t)$ are smooth functions). Then J is a Jacobi field if and only if $J_i'' + \sum_{j=1}^n R_{ij}J_j = 0$ for all $i = 1, \dots, n$.*

Corollary 9.7. *For any choice of $v, w \in T_{c(t_0)}M$ there exists a unique Jacobi field J along c such that $J(t_0) = v$, $\frac{D}{dt}J(t_0) = w$.*

Remark 9.8. Corollary 9.7 implies that for any geodesic $c(t)$ the vector space $J_c(M)$ of Jacobi fields along c has dimension $2n$. Moreover, the map $T_{c(t_0)}M \times T_{c(t_0)}M \to J_c(M)$ defined by $(v, w) \mapsto J$ s.t. $J(t_0) = v$, $\frac{D}{dt}J(t_0) = w$ is an isomorphism of vector spaces.

Lemma 9.9. *Let $c : [0,1] \to M$ be a geodesic and $J \in J_c(M)$ be a Jacobi field along c. Suppose $J(0) = 0$. Then there exists a geodesic variation F of c such that $J = \frac{\partial F}{\partial s}(0,t)$.*

9.2 Conjugate points and orthogonal Jacobi fields

Definition 9.10. Let $c : [a,b] \to M$ be a geodesic, $a \le t_0 < t_1 \le b$, $p = c(t_0)$, $q = c(t_1)$. The point q is conjugate to p along $c(t)$ if there exists a Jacobi field $J \in J_c(M)$, $J \not\equiv 0$ such that $J(t_0) = J(t_1) = 0$.

Example 9.11. On the sphere S^2 (with induced metric), the South pole is conjugate to the North pole along each geodesic passing through both these points.

Definition 9.12. A point $q \in M$ is conjugate to a point $p \in M$ if there exists a geodesic $c(t)$ passing through p and q such that q is conjugate to p along $c(t)$.

Definition 9.13. A multiplicity of a conjugate point $c(t_1)$ (with respect to a point $c(t_0)$) is the number of linear independent Jacobi fields along c such that $J(t_0) = J(t_1) = 0$, in other words, it is equal to dim $J_c^{t_0,t_1}(M)$, where $J_c^{t_0,t_1}(M) = \{J \in J_c(M) \mid J(t_0) = J(t_1) = 0\}$.

Remark 9.14. Multiplicity does not exceed $n-1$.

Lemma 9.15. *Let $J \in J_c(M)$ be a Jacobi field along a geodesic $c(t) = \exp_p tv$. Suppose $J(0) = 0$. Then there exist vectors $v, w \in T_{c(0)}M$ s.t. $J(t) = (D\exp_p)(tv)tw$. Here we identify $T_vT_{c(0)}M$ with $T_{c(0)}M$.*

Lemma 9.16. *A point $q = c(t_1)$ is conjugate to $p = c(0)$ along a geodesic $c(t) = \exp_p tv$ if and only if the point $v_1 = t_1 v \in T_pM$ is a critical point of the exponential map $\exp_p$ (i.e.* dim $\ker(D\exp_p)(t_1 v) > 0$*). Multiplicity of q is equal to* dim $\ker(D\exp_p)(t_1 v)$.

Lemma 9.17. *Let $c : [a,b] \to M$ be a geodesic, $a \le t_0 < t_1 \le b$. Suppose that $c(t_1)$ is not conjugate to $c(t_0)$. Take $v \in T_{c(t_0)}M$, $u \in T_{c(t_1)}M$. Then there exists a unique Jacobi field J along c s.t. $J(t_0) = v$, $J(t_1) = u$.*

Lemma 9.18. *Let $J \in J_c(M)$ be a Jacobi field along a geodesic $c(t)$. Then the function $t \mapsto \langle J(t), c'(t)\rangle$ is linear. More precisely, $\langle J(t), c'(t)\rangle = \langle J(0), c'(0)\rangle + t\langle \frac{D}{dt}J(0), c'(0)\rangle$.*

Corollary 9.19. *Let $\langle J(t_1), c'(t_1)\rangle = \langle J(t_2), c'(t_2)\rangle$. Then the function $t \mapsto \langle J(t), c'(t)\rangle$ is constant.*

Definition 9.20. A Jacobi field $J \in J_c(M)$ is orthogonal if $\langle J, c'\rangle \equiv 0$. The space of all orthogonal Jacobi fields along c is denoted by $J_c^{\perp}$.

Corollary 9.21. *(a) Let $J(0) = 0$. Then J is orthogonal if and only if $\langle \frac{D}{dt}J(0), c'(0)\rangle = 0$.*

(b) $dim J_c^{\perp} = 2n-2$.

(c) $dim J_c^{\perp,t_0} = n-1$, where $J_c^{\perp,t_0} = \{J \in J_c(M) \mid \langle J, c'\rangle \equiv 0, J(t_0) = 0\}$.

Example 9.22. Jacobi fields on $\mathbb{R}^2$.

Theorem 9.23. *Let c be a geodesic. Then every Jacobi field $J \in J_c(M)$ is a variational vector field for some geodesic variation $F(s,t)$ of c.*

9.3 Minimal geodesics and conjugate points

Theorem 9.24. *Let $c : [0,b] \to M$ be a geodesic and let $c(a)$ be a point conjugate to $c(0)$, $0 < a < b$. Then c is not a minimal geodesic between $c(0)$ and $c(b)$.*

Lemma 9.25, Corollary 9.26 and Lemma 9.27 serve to prove Theorem 9.24; we skip them here.

Example 9.28. No conjugate points on a flat torus.

Definition 9.29. Let c be a geodesic, $p = c(0)$. A point $q = c(t_0)$ is a cut point of p along c if the geodesic c is minimal on $[0, t_0]$ and is not minimal on $[0, t]$ for $t > t_0$.

A cut locus of p is the set of all cut points of p (with respect to all geodesics though p).

Example 9.30. Cut loci on the sphere S^2 and on a flat torus T^2.

Fact. If $c(t_0)$ is a cut point of $p = c(0)$ along c, then either

(a) $c(t_0)$ is the first conjugate point of $c(0)$ along c, or

(b) there exists a geodesic $\gamma \ne c$ from p to $c(t_0)$ such that $l(\gamma) = l(c)$.

Example 9.31. Basis of the space of Jacobi fields on hyperbolic plane.

9.4 Theorem of Cartan – Hadamard

Definition 9.32. A topological space is simply-connected if for each curve $c : [0,1] \to M$ with $c(0) = c(1)$ there exists a continuous map $F : [0,1] \times [0,1] \to M$ such that $F(1,t) = c(t)$, $F(0,t) = p$ for some $p \in M$, and $F(s,0) = F(s,1)$ for every $s \in [0,1]$.

Examples. $\mathbb{R}^n$ is simply-connected, S^n is simply-connected for $n > 1$; S^1 and T^n (torus) are not simply-connected.

Theorem 9.33 (Cartan – Hadamard)**.** *Let M be a complete connected simply-connected Riemannian manifold of non-positive sectional curvature. Then M is diffeomorphic to $\mathbb{R}^n$, where n is the dimension of M.*