Durham University
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Differential Geometry III, Solutions 1 (Week 11)

Isometries and conformal maps - 1

1.1. Let $a > 0$. Construct explicitly a local isometry from the plane $P = \{ (u, v, 0) \in \mathbb{R}^3 \mid u, v \in \mathbb{R} \}$ onto the cylinder $S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2 \}$.

Solution:

A canonical parametrization of the plane P is

$$\boldsymbol{x}\colon U = \mathbb{R}^2 \longrightarrow P, \qquad \boldsymbol{x}(u, v) = (u, v, 0).$$

Clearly, $\boldsymbol{x}_u = (1, 0, 0)$, $\boldsymbol{x}_v = (0, 1, 0)$ and $E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = 1$, $F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0$ and $G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle = 1$.

We define a candidate for the isometry via this parametrisation

$$f\colon P \longrightarrow S, \qquad f(u, v, 0) := (a\cos(\omega u), a\sin(\omega u), v)$$

for some positive constant $\omega > 0$ (we could also interchange the role of u and v) (more precisely, we define $f \circ \boldsymbol{x}\colon U \longrightarrow S$). In order to check that f is a local isometry, we just need to calculate the coefficents of the fundamental form of S with respect to the (local) parametrisation $f \circ \boldsymbol{x}$, and see whether they equal E, F and G. But here we have

$$f_u = (f \circ \boldsymbol{x})_u = (-a\omega\sin(\omega u), a\omega\cos(\omega u), 0) \quad \text{and} \quad f_v = (f \circ \boldsymbol{x})_v = (0, 0, 1),$$

so that

$$\widetilde{E} = \langle f_u, f_u \rangle = a^2\omega^2, \quad \widetilde{F} = \langle f_u, f_v \rangle = 0 \quad \text{and} \quad \widetilde{G} = \langle f_v, f_v \rangle = 1.$$

We have $\widetilde{F} = F$ and $\widetilde{G} = G$. In order to have $\widetilde{E} = E$, we need $\omega = 1/a$, then f is a local isometry (by Proposition 8.15).

1.2. ($\star$) Let b be a positive number such that $\sqrt{1 + b^2}$ is an integer n. Let S be the circular cone obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (v, 0, bv)$, $v > 0$, about the z-axis. Let the coordinate xy-plane P be parametrized by polar coordinates (r, ϑ):

$$\boldsymbol{x}\colon U = (0, \infty) \times (0, 2\pi) \longrightarrow P, \quad \boldsymbol{x}(r, \vartheta) = (r\cos\vartheta, r\sin\vartheta, 0).$$

Show that the map $f\colon P \setminus \{(0, 0, 0)\} \longrightarrow S$ defined on $\boldsymbol{x}(U)$ by

$$f(\boldsymbol{x}(r, \vartheta)) = \frac{1}{n}\big(r\cos n\vartheta, r\sin n\vartheta, br\big)$$

is a local isometry on $\boldsymbol{x}(U)$.

Solution:

We have

$$\boldsymbol{x}_r = (\cos\vartheta, \sin\vartheta, 0) \quad \text{and} \quad \boldsymbol{x}_\vartheta = (-r\sin\vartheta, r\cos\vartheta, 0),$$

so that the coefficients of the first fundamental form of P with respect to the parametrization $\boldsymbol{x}$ *(polar coordinates — parametrized $P \setminus \{\mathbf{0}\}$)* are

$$E(r, \vartheta) = 1, \qquad F(r, \vartheta) = 0 \quad \text{and} \quad G(r, \vartheta) = r^2.$$

Now calculate

$$f_r := (f \circ \boldsymbol{x})_r = \frac{1}{n}(\cos(n\vartheta), \sin(n\vartheta), b) \quad \text{and} \quad f_\vartheta := (f \circ \boldsymbol{x})_\vartheta = (-r\sin(n\vartheta), r\cos(n\vartheta), 0),$$

so that

$$\widetilde{E} := \langle f_r, f_r \rangle = \frac{1+b^2}{n^2}, \qquad \widetilde{F} := \langle f_r, f_\vartheta \rangle = 0 \quad \text{and} \quad \widetilde{G} := \langle f_\vartheta, f_\vartheta \rangle = r^2.$$

By assumption, $(1+b^2)/n^2 = 1$, so that $\widetilde{E} = E$, $\widetilde{F} = F$ and $\widetilde{G} = G$, hence f is a local isometry by Proposition 8.15.

1.3. Let S_1, S_2, S_3 be regular surfaces.

(a) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are local isometries. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a local isometry.

(b) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are conformal maps with conformal factors $\lambda\colon S_1 \longrightarrow (0,\infty)$ and $\mu\colon S_2 \longrightarrow (0,\infty)$, respectively. Prove that $g\circ f\colon S_1 \longrightarrow S_3$ is a conformal map and calculate its conformal factor. (The conformal factor of a conformal map is the function appearing as factor in front of the inner product in the definition.)

(c) Let f and g be conformal maps with conformal factors λ and μ as in the previous part. Find a condition on λ and μ such that $g \circ f$ is a *(local) isometry.*

Solution:

(a) By the definition of a local isometry,

$$\langle d_{p_1} f(\boldsymbol{v}_1), d_{p_1} f(\boldsymbol{w}_1)\rangle_{f(p_1)} = \langle \boldsymbol{v}_1, \boldsymbol{w}_1\rangle_{p_1} \quad \text{and} \quad \langle d_{p_2} g(\boldsymbol{v}_2), d_{p_2} f(\boldsymbol{w}_2)\rangle_{g(p_2)} = \langle \boldsymbol{v}_2, \boldsymbol{w}_2\rangle_{p_2}$$

for all $p_1 \in S_1$, $\boldsymbol{v}_1, \boldsymbol{w}_1 \in T_{p_1}S_1$ and $p_2 \in S_2$, $\boldsymbol{v}_2, \boldsymbol{w}_2 \in T_{p_2}S_2$.
This notation is also already part ot the solution: applying these two equations with $p_2 = f(p_1)$, $\boldsymbol{v}_2 = d_{p_1}f(\boldsymbol{v}_1)$ and $\boldsymbol{w}_2 = d_{p_1}f(\boldsymbol{w}_1)$, and using the chain rule

$$d_{p_1}(g \circ f)(\boldsymbol{w}_1) = d_{f(p_1)}g(d_{p_1}f(\boldsymbol{w}_1))$$

for all $p_1 \in S_1$ and $\boldsymbol{w}_1 \in T_{p_1}S_1$, we obtain

$$\begin{aligned}\langle d_{p_1}(g\circ f)(\boldsymbol{v}_1), d_{p_1}(g\circ f) f(\boldsymbol{w}_1)\rangle_{(g\circ f)(p_1)} &= \langle d_{f(p_1)}g(d_{p_1}(\boldsymbol{v}_1)), d_{f(p_1)}g(d_{p_1}(\boldsymbol{w}_1))\rangle_{(g(f(p_1))} \\ &= \langle d_{p_1}(\boldsymbol{v}_1), d_{p_1}(\boldsymbol{w}_1)\rangle_{f(p_1)} \\ &= \langle \boldsymbol{v}_1, \boldsymbol{w}_1\rangle_{p_1}\end{aligned}$$

using the chain rule for the first, the isometry of g for the second and the isometry of f for the last equality. Hence we have shown that $g \circ f)$ is a local isometry using the definition.

(b) The proof is almost the same as the one of the first part: since f and g are conformal maps, we have

$$\langle d_{p_1} f(\boldsymbol{v}_1), d_{p_1} f(\boldsymbol{w}_1)\rangle_{f(p_1)} = \lambda(p_1)\langle \boldsymbol{v}_1, \boldsymbol{w}_1\rangle_{p_1} \quad \text{and} \quad \langle d_{p_2} g(\boldsymbol{v}_2), d_{p_2} f(\boldsymbol{w}_2)\rangle_{g(p_2)} = \mu(p_2)\langle \boldsymbol{v}_2, \boldsymbol{w}_2\rangle_{p_2}$$

for all $p_1 \in S_1$, $\boldsymbol{v}_1, \boldsymbol{w}_1 \in T_{p_1}S_1$ and $p_2 \in S_2$, $\boldsymbol{v}_2, \boldsymbol{w}_2 \in T_{p_2}S_2$.
Applying these two equations with $p_2 = f(p_1)$, $\boldsymbol{v}_2 = d_{p_1}f(\boldsymbol{v}_1)$ and $\boldsymbol{w}_2 = d_{p_1}f(\boldsymbol{w}_1)$, and using again the chain rule we obtain

$$\begin{aligned}\langle d_{p_1}(g\circ f)(\boldsymbol{v}_1), d_{p_1}(g\circ f) f(\boldsymbol{w}_1)\rangle_{(g\circ f)(p_1)} &= \langle d_{f(p_1)}g(d_{p_1}(\boldsymbol{v}_1)), d_{f(p_1)}g(d_{p_1}(\boldsymbol{w}_1))\rangle_{(g(f(p_1))} \\ &= \mu(f(p_1))\langle d_{p_1}(\boldsymbol{v}_1), d_{p_1}(\boldsymbol{w}_1)\rangle_{f(p_1)} \\ &= \mu(f(p_1))\lambda(p_1)\langle \boldsymbol{v}_1, \boldsymbol{w}_1\rangle_{p_1}\end{aligned}$$

using the chain rule for the first, the conformality of g for the second and the conformality of f for the last equality. Hence we have shown that $g \circ f)$ is a conformal map with conformal factor

$$(\mu \circ f)\cdot \lambda\colon S_1 \longrightarrow (0,\infty, \qquad p_1 \mapsto \mu(f(p_1))\lambda(p_1).$$

(c) The third part is again rather trivial. We want that $(\mu \circ f) \cdot \lambda$ equals the constant function 1 on S_1, i.e., that

$$\mu(f(p_1)) = \frac{1}{\lambda(p_1)}$$

for all $p_1 \in S_1$. In particular, we do not need any restriction on the behaviour of μ outside the range $f(S_1)$ of f.

1.4. Let S be the surface of revolution parametrized by

$$\boldsymbol{x}(u,v) = \Big(\cos v \cos u, \cos v \sin u, -\sin v + \log \tan\Big(\frac{\pi}{4} + \frac{v}{2}\Big)\Big),$$

where $0 < u < 2\pi, 0 < v < \pi/2$. Let S_1 be the region

$$S_1 = \{\, \boldsymbol{x}(u,v) \,|\, 0 < u < \pi, 0 < v < \pi/2 \,\}$$

and let S_2 be the region

$$S_2 = \{\, \boldsymbol{x}(u,v) \,|\, 0 < u < 2\pi, \pi/3 < v < \pi/2 \,\}.$$

Show that the map

$$\boldsymbol{x}(u,v) \mapsto \boldsymbol{x}\Big(2u, \arccos\Big(\frac{1}{2}\cos v\Big)\Big)$$

is an isometry from S_1 onto S_2.

Solution:

The map $f\colon S_1 \longrightarrow S_2$ is actually a bijection (see below), so one can prove that it gives rise to a local parametrization; we will use Prop. 8.15 from the lectures and show that the coefficients E, F and G (w.r.t. the parametrization $\boldsymbol{x}$) are the same as the coefficients $\widetilde{E}$, $\widetilde{F}$ and $\widetilde{G}$ w.r.t the parametrization

$$\widetilde{\boldsymbol{x}}(u,v) := \boldsymbol{x}\Big(2u, \arccos\Big(\frac{1}{2}\cos v\Big)\Big).$$

Let us calculate E, F and G first. We have

$$\boldsymbol{x}_u = (-\cos v \sin u, \cos v \cos u, 0), \qquad \boldsymbol{x}_v = (-\sin v \cos u, -\sin v \sin u, -\cos v + 1/\cos v),$$

as the derivative of g with $g(v) = -\sin v + \log\tan(\pi/4 + v/2)$ is

$$\begin{aligned} g'(v) &= -\cos v + \frac{1}{2}\Big(\tan\Big(\frac{\pi}{4} + \frac{v}{2}\Big)\Big)^{-1} \tan'\Big(\frac{\pi}{4} + \frac{v}{2}\Big) \\ &= -\cos v + \frac{\cos(\pi/4 + v/2)}{2\sin(\pi/4 + v/2)\cos^2(\pi/4 + v/2)} \\ &= -\cos v + \frac{1}{\sin(\pi/2 + v)} \\ &= -\cos v + \frac{1}{\cos v} = \frac{-\cos^2 v + 1}{\cos v} = \frac{\sin^2 v}{\cos v}. \end{aligned}$$

In particular,

$$\begin{aligned} E(u,v) &= \cos^2 v, \qquad\qquad F(u,v) = 0, \\ G(u,v) &= \sin^2 v + \Big(\frac{1}{\cos v} - \cos v\Big)^2 \\ &= 1 - \cos^2 v + \frac{1}{\cos^2 v} - 2 + \cos^2 v \\ &= \frac{1}{\cos^2 v} - 1 = \frac{1 - \cos^2 v}{\cos^2 v} = \tan^2 v \end{aligned}$$

Let us now calculate the coefficients w.r.t. the parametrization $\widetilde{\boldsymbol{x}}$ (make sure you use the arguments of the functions correctly):

$$\begin{aligned} f_u(u,v) &= \widetilde{\boldsymbol{x}}_u(u,v) = 2\boldsymbol{x}_u(2u, \arccos(\cos v/2)) \\ f_v(u,v) &= \widetilde{\boldsymbol{x}}_v(u,v) = \varphi'(v)\boldsymbol{x}_v(2u, \arccos((\cos v)/2)) \\ &= \frac{\sin v}{2\sqrt{1-(\cos^2 v)/4}}\boldsymbol{x}_v(2u, \arccos((\cos v)/2)) \end{aligned}$$

since the derivative of φ given by $\varphi(v) = \arccos((\cos v)/2)$ is

$$\varphi'(v) = \frac{1}{2}(-\sin v)\arccos'((\cos v)/2) = \frac{\sin v}{2\sqrt{1-(\cos^2 v)/4}}.$$

In particular,

$$\begin{aligned} \langle f_u(u,v), f_u(u,v)\rangle &= \widetilde{E}(u,v) = 4E(2u, \arccos(\cos v/2)), \\ \langle f_u(u,v), f_v(u,v)\rangle &= \widetilde{F}(u,v) = 2\varphi'(v)F(\dots) = 0 \\ \langle f_v(u,v), f_v(u,v)\rangle &= \widetilde{G}(u,v) = \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}G(2u, \arccos(\cos v/2)). \end{aligned}$$

Let us now simplify these expressions in order to obtain $E = \widetilde{E}$ and $G = \widetilde{G}$ ($F = \widetilde{F} = 0$ is aready clear):

$$\begin{aligned} \widetilde{E}(u,v) &= 4E(2u, \arccos(\cos v/2)) \\ &= 4\cos^2 \arccos(\cos v/2)) \\ &= 4\big(\cos v/2\big)^2 = \cos^2 v = E(u,v), \end{aligned}$$

as $\cos(\arccos z) = z$ for $z \in [-1,1]$.

Moreover,

$$\begin{aligned} \widetilde{G}(u,v) &= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}G(2u, \arccos(\cos v/2)) \\ &= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}\Big(\frac{1}{\cos^2(\arccos(\cos v/2))} - 1\Big) \\ &= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}\Big(\frac{1}{\cos^2 v/4} - 1\Big) \\ &= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}\Big(\frac{1-\cos^2 v/4}{\cos^2 v/4}\Big) \\ &= \frac{\sin^2 v}{\cos^2 v} = G(u,v) \end{aligned}$$

(where we use the expression of $G(u,v)$ involving $\cos v$ only for the second equality).

Hence, by Proposition 8.15, f is a local isometry.

For f being an isometry, we also need that $f\colon S_1 \longrightarrow S_2$ is a bijection: Basically, we map $(u,v) \in U_1 = (0,\pi) \times (0,\pi/2)$ onto $\Phi(u,v) := (2u, \arccos((\cos v)/2)) \in U_2 = (0,2\pi) \times (\pi/3, \pi/2)$, and as

$$\psi\colon (0,\pi) \longrightarrow (0,2\pi), \quad \psi(u) = 2u$$

and

$$\varphi\colon (0,\pi/2) \longrightarrow (\pi/3,\pi/2), \quad \varphi(v) = \arccos((\cos v)/2)$$

are both bijections, $\Phi\colon U_1 \longrightarrow U_2$ is a bijection and hence also $f = \boldsymbol{x} \circ \Phi \circ \boldsymbol{x}^{-1}$.