Durham University
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Differential Geometry III, Solutions 2 (Week 12)

Isometries and conformal maps - 2

2.1. ($\star$) Let S be a surface of revolution. Prove that any rotation about the axis of revolution is an isometry of S.

Solution:

Let S be parametrised by $\boldsymbol{x}\colon U \longrightarrow S$ with

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

and $U = (-\pi,\pi)\times J$ or $U = (0,2\pi)\times J$, where $f\colon J \longrightarrow (0,\infty)$ and $g\colon J \longrightarrow \mathbb{R}$ are the functions of the generating curve given by $v \mapsto (f(v), 0, g(v))$. We know that

$$E(u,v) = f(v)^2, \qquad F(u,v) = 0, \qquad G(u,v) = f'(v)^2 + g'(v)^2$$

The rotation R by an angle ϑ around the symmetry axis is define by

$$R(\boldsymbol{x}(u,v)) = \boldsymbol{x}(u+\vartheta, v)$$

(for appropriate parameter values $(u,v)\in U$ such that $(u+\vartheta,v)\in U$). Then we have

$$\begin{aligned} R_u(u,v) &= (R\circ\boldsymbol{x})_u(u,v) = \boldsymbol{x}_u(u+\vartheta,v)\\ R_v(u,v) &= (R\circ\boldsymbol{x})_v(u,v) = \boldsymbol{x}_v(u+\vartheta,v), \end{aligned}$$

hence

$$\begin{aligned} \widetilde{E}(u,v) = \langle R_u(u,v), R_u(u,v)\rangle &= \boldsymbol{x}_u(u+\vartheta,v)\cdot\boldsymbol{x}_u(u+\vartheta,v) = E(u+\vartheta,v) = f(v)^2\\ &= E(u,v)\\ \widetilde{F}(u,v) = \langle R_u(u,v), R_v(u,v)\rangle &= \boldsymbol{x}_u(u+\vartheta,v)\cdot\boldsymbol{x}_v(u+\vartheta,v) = 0 = F(u,v)\\ \widetilde{G}(u,v) = \langle R_v(u,v), R_v(u,v)\rangle &= \boldsymbol{x}_v(u+\vartheta,v)\cdot\boldsymbol{x}_v(u+\vartheta,v)\\ &= G(u+\vartheta,v) = f'(v)^2 + g'(v)^2 = G(u,v) \end{aligned}$$

(in other words, the coefficients do not depend on the angle variable u).

Hence, f is a local isometry. Moreover, $R = R_\vartheta\colon S \longrightarrow S$ is obviously a bijection, so it is a global isometry.

Alternatively, one can note that $R = R_\vartheta\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear orthogonal map, so its differential $d_pR_\vartheta = R_\vartheta$ preserves lengths of all tangent (to $\mathbb{R}^3$) vectors. This means that R_ϑ is a global isometry of any surface onto its image. Now, since $R_\vartheta(S) = S$, R_ϑ is a global isometry of S.

2.2. The disc model of the hyperbolic plane.
Let $\mathbb{D}$ denote the unit disc $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$ with first fundamental form

$$\widetilde{E} = \widetilde{G} = \frac{4}{(1-x^2-y^2)^2}, \quad \widetilde{F} = 0.$$

Let $\mathbb{H}$ be the hyperbolic plane with coordinates $(u,v)\in\mathbb{R}\times(0,\infty)$ and first fundamental form

$$E = G = \frac{1}{v^2}, \quad F = 0.$$

Show that the map $\boldsymbol{f}\colon \mathbb{H} \longrightarrow \mathbb{D}$ given by

$$\boldsymbol{f}(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}, \qquad z = u + \mathrm{i}v \in \mathbb{H},$$

is an isometry.

Solution: We can consider $(x, y) = (\mathrm{Re}(\boldsymbol{f}), \mathrm{Im}(\boldsymbol{f}))$ as a coordinate system on $\mathbb{D}$ (the bijectivity can be checked easily, please also check that the differential is non-degenerate everywhere).

If we take a tangent vector $\boldsymbol{w} = (a, b) \in T_{(u,v)}\mathbb{H}$, then the square of its length is equal to

$$\langle \boldsymbol{w}, \boldsymbol{w}\rangle_{(u,v)} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2E + 2abF + b^2G = \frac{a^2 + b^2}{v^2} = \frac{\langle \boldsymbol{w}, \boldsymbol{w}\rangle_{\mathrm{Eucl}}}{v^2}$$

by the definition of the coefficients of the first fundamental form, where $\langle \boldsymbol{w}, \boldsymbol{w}\rangle_{\mathrm{Eucl}}$ is the Euclidean dot product.

The differential of $\boldsymbol{f}$ can be written as

$$d_{(u,v)}\boldsymbol{f} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{pmatrix}, = \begin{pmatrix} \boldsymbol{f}_u & \boldsymbol{f}_v \end{pmatrix},$$

where

$$\boldsymbol{f}_u = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \end{pmatrix} = d_{(u,v)}\boldsymbol{f}((1,0)), \qquad \boldsymbol{f}_v = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \end{pmatrix} = d_{(u,v)}\boldsymbol{f}((0,1)).$$

Then

$$d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}) = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a\boldsymbol{f}_u + b\boldsymbol{f}_v.$$

The square of the length of $d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})$ is then can be computed as

$$\langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})\rangle_{\boldsymbol{f}(u,v)} = (d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}))^T \begin{pmatrix} \widetilde{E}(u,v) & \widetilde{F}(u,v) \\ \widetilde{F}(u,v) & \widetilde{G}(u,v) \end{pmatrix} (d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})) =$$

$$\frac{4\langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})\rangle_{\mathrm{Eucl}}}{(1 - x^2 - y^2)^2} = \frac{4}{(1 - x^2 - y^2)^2}(a^2\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle_{\mathrm{Eucl}} + 2ab\langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle_{\mathrm{Eucl}} + b^2\langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle_{\mathrm{Eucl}}).$$

To show that $\boldsymbol{f}$ is an isometry, We need to show that $\langle \boldsymbol{w}, \boldsymbol{w}\rangle_{(u,v)} = \langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})\rangle_{\boldsymbol{f}(u,v)}$.

Writing

$$\begin{aligned} x + iy &= \boldsymbol{f}(u + iv) \\ &= \frac{u + iv - i}{u + iv + i} \\ &= \frac{(u + iv - i)(u - iv - i)}{u^2 + (v+1)^2} \\ &= \frac{u^2 + v^2 - 1}{u^2 + (v+1)^2} + i\frac{-2u}{u^2 + (v+1)^2}, \end{aligned}$$

we have

$$\boldsymbol{f}(u,v) = (x(u,v), y(u,v)) = \frac{1}{u^2 + (v+1)^2}(u^2 + v^2 - 1, -2u).$$

In particular, we can calculate that

$$1 - x^2 - y^2 = 1 - \frac{(u^2 + v^2 - 1)^2 + (-2u)^2}{(u^2 + (v+1)^2)^2} = \frac{4v}{u^2 + (v+1)^2}.$$

Taking partial derivatives gives

$$\begin{aligned} \boldsymbol{f}_u &= \frac{1}{(u^2 + (v+1)^2)^2}(4u(v+1), 2u^2 - 2(v+1)^2), \\ \boldsymbol{f}_v &= \frac{1}{(u^2 + (v+1)^2)^2}(-2u^2 + 2(v+1)^2, 4u(v+1)). \end{aligned}$$

Computing the (Euclidean) inner products of the vectors above, we obtain

$$\begin{aligned}
\boldsymbol{f}_u \cdot \boldsymbol{f}_u &= \frac{4}{(u^2+(v+1)^2)^4}(4u^2(v+1)^2 + (u^2-(v+1)^2)^2) = \frac{4}{(u^2+(v+1)^2)^2}, \\
\boldsymbol{f}_u \cdot \boldsymbol{f}_v &= 0, \\
\boldsymbol{f}_v \cdot \boldsymbol{f}_v &= \frac{4}{(u^2+(v+1)^2)^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
4\frac{\boldsymbol{f}_u \cdot \boldsymbol{f}_u}{(1-x^2-y^2)^2} = 4\frac{\frac{4}{(u^2+(v+1)^2)^2}}{\left(\frac{4v}{u^2+(v+1)^2}\right)^2} = \frac{1}{v^2} &= E, \\
4\frac{\boldsymbol{f}_u \cdot \boldsymbol{f}_v}{(1-x^2-y^2)^2} = 0 &= F, \\
4\frac{\boldsymbol{f}_v \cdot \boldsymbol{f}_v}{(1-x^2-y^2)^2} = \frac{1}{v^2} &= G,
\end{aligned}$$

and thus

$$\langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})\rangle_{\boldsymbol{f}(u,v)} = a^2E + 2abF + b^2G = \langle \boldsymbol{w}, \boldsymbol{w}\rangle_{(u,v)}$$

(compare to Proposition 8.15 from the lectures).

2.3. Hyperboloid model of the hyperbolic plane.
Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form defined by

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3$$

(the quadratic space $(\mathbb{R}^3, Q)$ is usually denoted by $\mathbb{R}^{2,1}$). Let

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, Q(x_1, x_2, x_3) = -1\}$$

(i.e. S is a hyperboloid of two sheets).

Recall that the *induced quadratic form* $I_{\boldsymbol{p}}$ on each tangent plane $T_{\boldsymbol{p}}S$ is defined by $I_{\boldsymbol{p}}(\boldsymbol{w}) = Q(\boldsymbol{w})$ for every $\boldsymbol{w} \in T_{\boldsymbol{p}}(S)$. Show that $I_{\boldsymbol{p}}$ is positive definite and that the map $f : \mathbb{D} \to S$ from the disc model of the hyperbolic plane (see the previous exercise) defined by

$$\boldsymbol{f}(x, y) = \frac{1}{1-x^2-y^2}(2x, 2y, 1+x^2+y^2), \qquad (x, y) \in \mathbb{D},$$

maps $\mathbb{D}$ isometrically onto the component of S for which $x_3 > 0$.

Solution:

Note that $\boldsymbol{f}$ is parametrization of the "upper" part of S (please check bijectivity!). In particular,

$$\begin{aligned}
\boldsymbol{f}_x &= \frac{2}{(1-x^2-y^2)^2}((1+x^2-y^2), 2xy, 2x), \\
\boldsymbol{f}_y &= \frac{2}{(1-x^2-y^2)^2}(2xy, (1-x^2+y^2), 2y),
\end{aligned}$$

which implies that

$$\begin{aligned}
\widetilde{E} &= Q(\boldsymbol{f}_x) = \frac{4}{(1-x^2-y^2)^4}((1+x^2-y^2)^2 + (2xy)^2 - (2x)^2) = \frac{4}{(1-x^2-y^2)^2} = E, \\
\widetilde{F} &= 0, \\
\widetilde{G} &= Q(\boldsymbol{f}_y) = \frac{4}{(1-x^2-y^2)^2} = G.
\end{aligned}$$