Durham University
Pavel Tumarkin

Epiphany 2017

Differential Geometry III, Homework 2 (Week 12)

Due date for starred problems: **Thursday, February 2**.

Isometries and conformal maps - 2

2.1. ($\star$) Let S be a surface of revolution. Prove that any rotation about the axis of revolution is an isometry of S.

2.2. The disc model of the hyperbolic plane.
Let $\mathbb{D}$ denote the unit disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with first fundamental form

$$\widetilde{E} = \widetilde{G} = \frac{4}{(1 - x^2 - y^2)^2}, \quad \widetilde{F} = 0.$$

Let $\mathbb{H}$ be the hyperbolic plane with coordinates $(u, v) \in \mathbb{R} \times (0, \infty)$ and first fundamental form

$$E = G = \frac{1}{v^2}, \quad F = 0.$$

Show that the map $f \colon \mathbb{H} \longrightarrow \mathbb{D}$ given by

$$f(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}, \qquad z = u + \mathrm{i}v \in \mathbb{H},$$

is an isometry.

2.3. Hyperboloid model of the hyperbolic plane.
Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form defined by

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3$$

(the quadratic space $(\mathbb{R}^3, Q)$ is usually denoted by $\mathbb{R}^{2,1}$). Let

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid Q(x_1, x_2, x_3) = -1\}$$

(i.e. S is a hyperboloid of two sheets).

Recall that the *induced quadratic form* $I_{\boldsymbol{p}}$ on each tangent plane $T_{\boldsymbol{p}}S$ is defined by $I_{\boldsymbol{p}}(\boldsymbol{w}) = Q(\boldsymbol{w})$ for every $\boldsymbol{w} \in T_{\boldsymbol{p}}(S)$. Show that $I_{\boldsymbol{p}}$ is positive definite and that the map $f : \mathbb{D} \to S$ from the disc model of the hyperbolic plane (see the previous exercise) defined by

$$f(x, y) = \frac{1}{1 - x^2 - y^2}\,(2x, 2y, 1 + x^2 + y^2), \qquad (x, y) \in \mathbb{D},$$

maps $\mathbb{D}$ isometrically onto the component of S for which $x_3 > 0$.