Durham University
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Epiphany 2017

Differential Geometry III, Homework 3 (Week 13)

Due date for starred problems: **Friday, February 17**.

Weingarten map, Gauss, mean and principal curvatures - 1

3.1. A local parametrization $\boldsymbol{x}$ of a surface S in $\mathbb{R}^3$ is called *orthogonal* provided $F = 0$ (so $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are orthogonal at each point). It is called *principal* if $F = 0$ and $M = 0$, where E, F, G (resp. L, M, N) are the coefficients of the first (resp. second) fundamental form.

(a) Let $\boldsymbol{x}$ be an *orthogonal* parametrization. Show that, at any point $p = \boldsymbol{x}(u, v)$ on S,

$$-d\boldsymbol{N}_p(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u + \frac{M}{G}\boldsymbol{x}_v, \qquad -d\boldsymbol{N}_p(\boldsymbol{x}_v) = \frac{M}{E}\boldsymbol{x}_u + \frac{N}{G}\boldsymbol{x}_v,$$

where $\boldsymbol{N}$ denotes the Gauss map.

(b) Assume now that the parametrization is *principal*. Show that $\kappa_1 = L/E$ and $\kappa_2 = N/G$ are the principal curvatures. Calculate the Gauss and mean curvature in terms of E, G, L, N. Determine the principal directions.

3.2. Calculation of the Weingarten map directly for surfaces of revolution

Let $f: J \longrightarrow (0, \infty)$ and $g: J \longrightarrow \mathbb{R}$ be smooth functions on some open interval J in $\mathbb{R}$ and let $\alpha: J \longrightarrow \mathbb{R}^3$ be a space curve given by $\alpha(v) = (f(v), 0, g(v))$. Assume that this curve is parametrized by arc length. Let S be the surface of revolution obtained by rotating α around the z-axis.

(a) Find suitable parametrizations $\boldsymbol{x}: U_i \longrightarrow S$ of S and determine parameter domains U_1 and U_2 covering the whole surface S. Calculate the normal vector $\boldsymbol{N}$ at $\boldsymbol{x}(u, v)$

(b) Express $a, b, c, d \in \mathbb{R}$ in $-d\boldsymbol{N}_p(\boldsymbol{x}_u) = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ and $-d\boldsymbol{N}_p(\boldsymbol{x}_v) = c\boldsymbol{x}_u + d\boldsymbol{x}_v$ in terms of f and g.

(c) Calculate the principal directions and principal curvatures.

(d) Calculate the Gauss and mean curvatures.

(e) Compare your results with Example 9.13 from the lectures.

3.3. Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$z = \frac{1}{1 + x^2 + y^2}.$$

Find the principal curvatures and the umbilic points (i.e., the points where the principal curvatures are the same). Give a sketch of the surface showing the regions of the surface where the Gauss curvature K is strictly positive and strictly negative.

3.4. ($\star$) **The pseudosphere**

The pseudosphere is the surface of revolution obtained by rotating the tractrix with parametrization $\boldsymbol{\alpha}(s) = (1/\cosh s, 0, s - \tanh s)$ around the z-axis. Prove that the pseudosphere has constant Gauss curvature $K = -1$.