Durham University
Pavel Tumarkin

Epiphany 2017

Differential Geometry III, Solutions 4 (Week 14)

Weingarten map, Gauss, mean and principal curvatures - 2

4.1. Let S be the surface given by the graph of the function $f\colon U \longrightarrow \mathbb{R}$ ($U \subset \mathbb{R}^2$ open). Calculate the Gauss and mean curvature of S in terms of f and its derivatives.

Solution: We choose the standard parametrization for a graph of a function, i.e.,

$$\boldsymbol{x}\colon U \longrightarrow S, \qquad \boldsymbol{x}(u,v) = (u, v, f(u,v)),$$

where $S = \{\, (u, v, f(u,v)) \,|\, (u,v) \in U \,\}$. Then we have

$$\boldsymbol{x}_u = (1,0,f_x), \qquad \boldsymbol{x}_v = (0,1,f_y), \qquad \boldsymbol{x}_u \times \boldsymbol{x}_v = (-f_x, -f_y, 1),$$
$$\boldsymbol{x}_{uu} = (0,0,f_{xx}), \qquad \boldsymbol{x}_{uv} = (0,0,f_{xy}), \qquad \boldsymbol{x}_{vv} = (0,0,f_{yy}).$$

From this, we see that the normal vector is

$$\boldsymbol{N} = \frac{1}{D}, \qquad D = \sqrt{1 + f_x^2 + f_y^2}$$

and we easily see that

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + f_x^2, \qquad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = f_x f_y, \qquad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + f_y^2,$$
$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{f_{xx}}{D}, \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = \frac{f_{xy}}{D}, \qquad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{f_{yy}}{D}.$$

Note that we have

$$EG - F^2 = (1+f_x^2)(1+f_y^2) - f_x^2 f_y^2 = 1 + f_x^2 + f_y^2 = \|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = D^2.$$

(observe that the equality $EG - F^2 = \|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2$ is always true, what is the geometrical meaning of this?)
Now, the Gauss curvature is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{f_{xx} f_{yy} - f_{xy}^2}{D^4} = \frac{\det H(f)}{D^4}, \quad \text{where} \quad H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

is the Hessian matrix of f. Moreover, the mean curvature is given by

$$H = \frac{EN - 2FM + GL}{EG - F^2} = \frac{(1+f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (f_y^2 + 1) f_{xx}}{D^3}.$$

4.2. ($\star$) **Enneper's surface**

Consider the surface in $\mathbb{R}^3$ parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\Big), \qquad (u,v) \in \mathbb{R}^2.$$

Show that

(a) the coefficients of the first and second fundamental forms are given by

$$E(u,v) = G(u,v) = (1 + u^2 + v^2)^2, \; F(u,v) = 0 \qquad \text{and} \qquad L = 2, \; M = 0, \; N = -2;$$

(b) the principal curvatures at $p = \boldsymbol{x}(u,v)$ are given by

$$\kappa_1(p) = \frac{2}{(1+u^2+v^2)^2}, \qquad \kappa_2(p) = -\frac{2}{(1+u^2+v^2)^2}.$$

Solution:

(a) We have

$$\boldsymbol{x}_u(u,v) = (1-u^2+v^2, 2uv, 2u), \qquad \boldsymbol{x}_v(u,v) = (2uv, 1+u^2-v^2, -2v)$$

so that the coefficients of the first fundamental form are

$$\begin{aligned} E(u,v) &= (1-u^2+v^2)^2 + 4uv^2 + 4u^2 = (1+u^2+v^2)^2, \\ F(u,v) &= 2uv(1-u^2+v^2) + 2uv(1+u^2-v^2) - 4uv = 0 \\ G(u,v) &= 4u^2v^2 + (1+u^2-v^2)^2 + 4v^2 = (1+u^2+v^2)^2 \end{aligned}$$

as desired. Moreover, we have

$$\boldsymbol{x}_{uu}(u,v) = (-2u, 2v, 2), \qquad \boldsymbol{x}_{uv}(u,v) = (2v, 2u, 0), \qquad \boldsymbol{x}_{vv}(u,v) = (2u, -2v, -2)$$

and

$$\begin{aligned} \boldsymbol{x}_u(u,v) \times \boldsymbol{x}_v(u,v) &= \begin{pmatrix} 1-u^2+v^2 \\ 2uv \\ 2u \end{pmatrix} \times \begin{pmatrix} 1+u^2-v^2 \\ 2uv \\ 2v \end{pmatrix} \\ &= \begin{pmatrix} -2u(1+u^2+v^2) \\ 2v(1+u^2+v^2) \\ (1-u^2-v^2)(1+u^2+v^2) \end{pmatrix} = (1+u^2+v^2) \begin{pmatrix} -2u \\ 2v \\ 1-u^2-v^2 \end{pmatrix} \end{aligned}$$

and $\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = EG - F^2 = (1+u^2+v^2)^4$, so that the normal vector is

$$\boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{1}{1+u^2+v^2} \begin{pmatrix} -2u \\ 2v \\ 1-u^2-v^2 \end{pmatrix}.$$

In particular, the coefficients of the second fundamental form are

$$\begin{aligned} L(u,v) &= \boldsymbol{x}_{uu} \cdot \boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{4u^2+4v^2+2(1-u^2-v^2)}{1+u^2+v^2} = 2 \\ M(u,v) &= \boldsymbol{x}_{uv} \cdot \boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{-4uv+4uv}{1+u^2+v^2} = 0 \\ N(u,v) &= \boldsymbol{x}_{uu} \cdot \boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{-4u^2-4v^2-2(1-u^2-v^2)}{1+u^2+v^2} = -2 \end{aligned}$$

again as desired.

(b) Let us first find the Gauss and mean curvature:

$$\begin{aligned} K &= \frac{LN-M^2}{EG-F^2} = \frac{-4}{(1+u^2+v^2)^4} \quad \text{and} \\ H &= \frac{EN-2FM+GL}{EG-F^2} = \frac{(-2+2)(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = 0 \end{aligned}$$

hence the principal curvatures are the solutions of $\kappa^2 - 2H\kappa + K = 0$, i.e., of

$$\kappa^2 = \frac{4}{(1+u^2+v^2)^4}, \qquad \text{or} \qquad \kappa = \pm\frac{2}{(1+u^2+v^2)^2},$$

as desired.

Remark. Note that the mean curvature of the Enneper surface S vanishes, so it is a minimal surface.

4.3. If S is a surface in $\mathbb{R}^3$ then a *parallel surface* to S is a surface $\widetilde{S}$ given by a local parametrization of the form

$$\boldsymbol{y}(u,v) = \boldsymbol{x}(u,v) + a\boldsymbol{N}(u,v), \qquad (u,v) \in U,$$

where $\boldsymbol{x}\colon U \longrightarrow S$ is a local parametrization of S, $\boldsymbol{N}\colon U \longrightarrow S^2$ the Gauss map in that parametrization, and a is some given constant.

(a) Show that

$$\boldsymbol{y}_u \times \boldsymbol{y}_v = (1 - 2Ha + Ka^2)\, \boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where H and K are the mean and Gauss curvatures of S.

(b) Assuming that $1 - 2Ha + Ka^2$ is never zero on S, show that the Gauss curvature $\widetilde{K}$ and mean curvature $\widetilde{H}$ of $\widetilde{S}$ are given by

$$\widetilde{K} = \frac{K}{1 - 2Ha + Ka^2}, \qquad \widetilde{H} = \frac{H - Ka}{1 - 2Ha + Ka^2}.$$

(c) If S has constant mean curvature $H \equiv c \neq 0$ and the Gauss curvature K is nowhere vanishing, show that the parallel surface given by $a = 1/(2c)$ has constant Gauss curvature $4c^2$.

Solution:

(a) First, note that

$$\boldsymbol{y}_u = \boldsymbol{x}_u + a\boldsymbol{N}_u, \qquad \text{and} \qquad \boldsymbol{y}_v = \boldsymbol{x}_u + a\boldsymbol{N}_v.$$

In order to express $\boldsymbol{y}_u \times \boldsymbol{y}_v$ in the desired form, it is helpful to express the derivative with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$:

$$\begin{aligned} -\boldsymbol{N}_u &= -(\boldsymbol{N} \circ x)_u = -d_p\boldsymbol{N}(\boldsymbol{x}_u) = A\boldsymbol{x}_u + B\boldsymbol{x}_v \quad \text{and} \\ -\boldsymbol{N}_v &= -(\boldsymbol{N} \circ x)_v = -d_p\boldsymbol{N}(\boldsymbol{x}_v) = C\boldsymbol{x}_u + D\boldsymbol{x}_v \end{aligned}$$

This is useful as we can express easily the Gauss and mean curvatures as the determinant and trace in terms of these coefficients as

$$K = AD - BC \qquad \text{and} \qquad H = \frac{A + D}{2}.$$

Now,

$$\begin{aligned} \boldsymbol{y}_u &= \boldsymbol{x}_u + a\boldsymbol{N}_u = \boldsymbol{x}_u + a(\boldsymbol{N} \circ \boldsymbol{x})_u = (1 - aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v \quad \text{and} \\ \boldsymbol{y}_v &= \boldsymbol{x}_v + a\boldsymbol{N}_v = \boldsymbol{x}_v + a(\boldsymbol{N} \circ \boldsymbol{x})_v = -aC\boldsymbol{x}_u + (1 - aD)\boldsymbol{x}_v \end{aligned}$$

and therefore

$$\begin{aligned} \boldsymbol{y}_u \times \boldsymbol{y}_v &= \big((1 - aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v\big) \times \big(-aC\boldsymbol{x}_u + (1 - aD)\boldsymbol{x}_v\big) \\ &= \big((1 - aA)(1 - aD) - a^2BC\big)\boldsymbol{x}_u \times \boldsymbol{x}_v \\ &= \big(1 - a(A + D) + a^2(AD - BC)\big)\boldsymbol{x}_u \times \boldsymbol{x}_v \\ &= \underbrace{(1 - 2Ha + Ka^2)}_{=:P}\, \boldsymbol{x}_u \times \boldsymbol{x}_v \end{aligned}$$

using the antisymmetry of the vector product ($\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$), and we obtain the desired formula.

(b) If $P := 1 - 2Ha + Ka^2 \neq 0$, then $\boldsymbol{y}_u \times \boldsymbol{y}_v$ is not vanishing, the normal vectors of S and $\widetilde{S}$ fulfil

$$\widetilde{\boldsymbol{N}} \circ \boldsymbol{y} = \boldsymbol{N} \circ \boldsymbol{x},$$

as $\boldsymbol{y}_u \times \boldsymbol{y}_v$ and $\boldsymbol{x}_u \times \boldsymbol{x}_v$ point in the same direction by the first part and the condition on $1 - 2Ha + Ka^2$.
Remark. Be careful with the statement $\widetilde{\boldsymbol{N}} = \boldsymbol{N}$, as the parametrisation is lost in this espression. This becomes important when taking derivatives (see below).

Let us use the same trick as for the surface S also for $\widetilde{S}$:

$$-\widetilde{\boldsymbol{N}}_u = -(\widetilde{\boldsymbol{N}} \circ y)_u = -d_p\widetilde{\boldsymbol{N}}(\boldsymbol{y}_u) = \widetilde{A}\boldsymbol{y}_u + \widetilde{B}\boldsymbol{y}_v \quad \text{and}$$
$$-\widetilde{\boldsymbol{N}}_v = -(\widetilde{\boldsymbol{N}} \circ y)_v = -d_p\widetilde{\boldsymbol{N}}(\boldsymbol{y}_v) = \widetilde{C}\boldsymbol{y}_u + \widetilde{D}\boldsymbol{y}_v.$$

Similarly as above, we have

$$\widetilde{K} = \widetilde{A}\widetilde{D} - \widetilde{B}\widetilde{C} \qquad \text{and} \qquad \widetilde{H} = \frac{\widetilde{A} + \widetilde{D}}{2}.$$

Taking the derivative of the equation $\widetilde{\boldsymbol{N}} \circ \boldsymbol{y} = \boldsymbol{N} \circ \boldsymbol{x}$ and combining the previous results gives

$$\begin{aligned}
A\boldsymbol{x}_u + B\boldsymbol{x}_v = -(\boldsymbol{N} \circ \boldsymbol{x})_u &= -(\widetilde{\boldsymbol{N}} \circ \boldsymbol{y})_u \\
&= \widetilde{A}\boldsymbol{y}_u + \widetilde{B}\boldsymbol{y}_v \\
&= \widetilde{A}\big((1-aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v\big) + \widetilde{B}\big(-aC\boldsymbol{x}_u + (1-aD)\boldsymbol{x}_v\big) \\
&= \big(\widetilde{A}(1-aA) - \widetilde{B}aC\big)\boldsymbol{x}_u + \big(-\widetilde{A}aB + \widetilde{B}(1-aD)\big)\boldsymbol{x}_v.
\end{aligned}$$

Comparing the coefficients gives the linear system

$$\begin{pmatrix} 1-aA & -aC \\ -aB & 1-aD \end{pmatrix} \begin{pmatrix} \widetilde{A} \\ \widetilde{B} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

for $(\widetilde{A}, \widetilde{B})$. The determinant of the coefficient matrix is

$$(1-aA)(1-aD) - a^2BC = 1 - (A+D)a + (AD-BC)a^2 = 1 - 2Ha + Ka^2 = P \neq 0,$$

so that we can take the inverse and obtain

$$\begin{pmatrix} \widetilde{A} \\ \widetilde{B} \end{pmatrix} = \frac{1}{P}\begin{pmatrix} 1-aD & aC \\ aB & 1-aA \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{P}\begin{pmatrix} (1-aD)A + aCB \\ aBA + (1-aA)B \end{pmatrix} = \frac{1}{P}\begin{pmatrix} A - aK \\ B \end{pmatrix}.$$

Similarly, we have (taking the derivative w.r.t. v) that

$$\begin{aligned}
C\boldsymbol{x}_u + D\boldsymbol{x}_v = -(\boldsymbol{N} \circ \boldsymbol{x})_v &= -(\widetilde{\boldsymbol{N}} \circ \boldsymbol{y})_v \\
&= \widetilde{C}\boldsymbol{y}_u + \widetilde{D}\boldsymbol{y}_v \\
&= \widetilde{C}\big((1-aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v\big) + \widetilde{D}\big(-aC\boldsymbol{x}_u + (1-aD)\boldsymbol{x}_v\big) \\
&= \big(\widetilde{C}(1-aA) - \widetilde{D}aC\big)\boldsymbol{x}_u + \big(-\widetilde{C}aB + \widetilde{D}(1-aD)\big)\boldsymbol{x}_v.
\end{aligned}$$

Comparing the coefficients gives the linear system

$$\begin{pmatrix} 1-aA & -aC \\ -aB & 1-aD \end{pmatrix} \begin{pmatrix} \widetilde{C} \\ \widetilde{D} \end{pmatrix} = \begin{pmatrix} C \\ D \end{pmatrix}$$

for $(\widetilde{B}, \widetilde{D})$, and as above, we obtain

$$\begin{pmatrix} \widetilde{C} \\ \widetilde{D} \end{pmatrix} = \frac{1}{P}\begin{pmatrix} 1-aD & aC \\ aB & 1-aA \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{P}\begin{pmatrix} (1-aD)C + aCD \\ aBC + (1-aA)D \end{pmatrix} = \frac{1}{P}\begin{pmatrix} C \\ D - aK \end{pmatrix}.$$

Now, we have

$$\widetilde{H} = \frac{1}{2}(\widetilde{A} + \widetilde{D}) = \frac{1}{2P}(A - aK + D - aK) = \frac{1}{P}(H - aK) = \frac{H - aK}{1 - 2aH + a^2K}$$

and

$$\begin{aligned}
\widetilde{K} = \widetilde{A}\widetilde{D} - \widetilde{B}\widetilde{C} &= \frac{1}{P^2}\big((A-aK)(D-aK) - BC\big) \\
&= \frac{1}{P^2}\big(\underbrace{AD - BC}_{=K} - a(A+D)K + a^2K\big) \\
&= \frac{K(1 - 2aH + a^2K)}{(1 - 2aH + a^2K)^2} = \frac{K}{1 - 2aH + a^2K}
\end{aligned}$$

as claimed.

(c) If S has constant mean curvature $H = c \neq 0$ and $K \neq 0$, then

$$\widetilde{K} = \frac{K}{1 - 2aH + a^2K} = \frac{K}{1 - 2c/2c + K/4c^2} = \frac{4c^2K}{K} = 4c^2$$

(and we have $P = 1 - 2aH + a^2K = K/4c^2 \neq 0$ as $K \neq 0$).

4.4. Let f be a smooth real-valued function defined on a connected open subset U of $\mathbb{R}^2$.

(a) Show that the graph S of f is a *minimal surface* in $\mathbb{R}^3$ (i.e., its mean curvature H vanishes) if and only if

$$f_{yy}(1 + f_x^2) - 2f_x f_y f_{xy} + f_{xx}(1 + f_y^2) = 0.$$

(b) Deduce that if $f(x, y) = g(x)$ then S is minimal if and only if S is a plane with normal vector parallel to the (x, z)-plane but not parallel to the x-axis.

(c) If $f(x, y) = g(x) + h(y)$, find the most general form of f in order for S to be minimal.
Hint: Use separation of variables

Solution:

(a) Let us take the formulae for the mean curvature of a surface which is a graph of a function from Exercise 4.1 (feel free to repeat the calculations, it is a good exercise). We have

$$H = \frac{EN - 2FM + GL}{EG - F^2} = \frac{(1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} + (f_y^2 + 1)f_{xx}}{D^3}$$

where $D = (1 + f_x^2 + f_y^2)^{1/2}$. In particular, a surface is a minimal surface iff

$$(1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} + (f_y^2 + 1)f_{xx} = 0,$$

as desired.

(b) If $f(x, y) = g(x)$, then $f_x = g'$, $f_y = 0$, and the equation $H = 0$ becomes just $g'' = 0$ (only the third summand is non-zero). In particular, $g(x) = ax + b$ for some constants $a, b \in \mathbb{R}$, i.e., f is the graph of a plane, and the normal vector of this plane is proportional to $(-a, 0, 1)$, i.e., parallel to the (x, z)-plane, but not to the x-axis (as the z-component is never 0).

(c) If $f(x, y) = g(x) + h(y)$, we obtain

$$f_x = g', \qquad f_y = h', \qquad f_{xx} = g'', \qquad f_{xy} = 0, \qquad f_{yy} = h'',$$

so that the equation $H = 0$ becomes

$$(1 + g'^2)h''(h'^2 + 1)g'' = 0, \quad \text{i.e.} \quad \frac{g''}{1 + g'^2} = -\frac{h''}{h'^2 + 1}$$

(separation of variables). Now, since the LHS depends on x only, while the RHS depends on y only, we have

$$\frac{g''}{g'^2 + 1} = c_0$$

for some constant c_0. Integrating gives (substituting $s = g'(x)$, i.e., $\mathrm{d}s = g''(x)\,\mathrm{d}x$)

$$\int \frac{1}{s^2 + 1}\,\mathrm{d}s = c_0 x + c_1, \quad \text{i.e.} \quad \arctan g'(x) = c_0 x + c_1 \quad \text{or} \quad g'(x) = \tan(c_0 x + c_1).$$

Integrating gives $g(x) = \log|\cos(c_0 x + c_1)|/c_0 + c_2$.
Similarly, $h(y) = -\log|\cos(-c_0 y + c_3)|/c_0 + c_4$. So the most general form of f is

$$\begin{aligned} f(x, y) &= \frac{1}{c_0}\log|\cos(c_0 x + c_1)| - \frac{1}{c_0}\log|\cos(-c_0 y + c_3)|/c_0 + c_5 \\ &= \frac{1}{c_0}\log\left|\frac{\cos(c_0 x + c_1)}{\cos(-c_0 y + c_3)}\right| + c_5 \end{aligned}$$

where c_0, c_1, c_3, c_5 are constants.