Durham University
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Epiphany 2017

Differential Geometry III, Homework 4 (Week 14)

Due date for starred problems: **Friday, February 17**.

Weingarten map, Gauss, mean and principal curvatures - 2

4.1. Let S be the surface given by the graph of the function $f\colon U \longrightarrow \mathbb{R}$ ($U \subset \mathbb{R}^2$ open). Calculate the Gauss and mean curvature of S in terms of f and its derivatives.

4.2. ($\star$) **Enneper's surface**

Consider the surface in $\mathbb{R}^3$ parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\Big), \qquad (u,v) \in \mathbb{R}^2.$$

Show that

(a) the coefficients of the first and second fundamental forms are given by

$$E(u,v) = G(u,v) = (1+u^2+v^2)^2,\ F(u,v) = 0 \qquad \text{and} \qquad L = 2,\ M = 0,\ N = -2;$$

(b) the principal curvatures at $p = \boldsymbol{x}(u,v)$ are given by

$$\kappa_1(p) = \frac{2}{(1+u^2+v^2)^2}, \qquad \kappa_2(p) = -\frac{2}{(1+u^2+v^2)^2}.$$

4.3. If S is a surface in $\mathbb{R}^3$ then a *parallel surface* to S is a surface $\widetilde{S}$ given by a local parametrization of the form

$$\boldsymbol{y}(u,v) = \boldsymbol{x}(u,v) + a\boldsymbol{N}(u,v), \qquad (u,v) \in U,$$

where $\boldsymbol{x}\colon U \longrightarrow S$ is a local parametrization of S, $\boldsymbol{N}\colon U \longrightarrow S^2$ the Gauss map in that parametrization, and a is some given constant.

(a) Show that

$$\boldsymbol{y}_u \times \boldsymbol{y}_v = (1 - 2Ha + Ka^2)\, \boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where H and K are the mean and Gauss curvatures of S.

(b) Assuming that $1 - 2Ha + Ka^2$ is never zero on S, show that the Gauss curvature $\widetilde{K}$ and mean curvature $\widetilde{H}$ of $\widetilde{S}$ are given by

$$\widetilde{K} = \frac{K}{1 - 2Ha + Ka^2}, \qquad \widetilde{H} = \frac{H - Ka}{1 - 2Ha + Ka^2}.$$

(c) If S has constant mean curvature $H \equiv c \neq 0$ and the Gauss curvature K is nowhere vanishing, show that the parallel surface given by $a = 1/(2c)$ has constant Gauss curvature $4c^2$.

4.4. Let f be a smooth real-valued function defined on a connected open subset U of $\mathbb{R}^2$.

(a) Show that the graph S of f is a *minimal surface* in $\mathbb{R}^3$ (i.e., its mean curvature H vanishes) if and only if

$$f_{yy}(1 + f_x^2) - 2f_xf_yf_{xy} + f_{xx}(1 + f_y^2) = 0.$$

(b) Deduce that if $f(x,y) = g(x)$ then S is minimal if and only if S is a plane with normal vector parallel to the (x,z)-plane but not parallel to the x-axis.

(c) If $f(x,y) = g(x) + h(y)$, find the most general form of f in order for S to be minimal.
Hint: Use separation of variables