Durham University
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Epiphany 2017

Differential Geometry III, Homework 5 (Week 15)

Due date for starred problems: **Thursday, March 2**.

Christoffel symbols and Gauss' Theorema Egregium

5.1. Show that the Gauss curvature K of the surface of revolution locally parametrized by

$$\boldsymbol{x}(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v)), \qquad (u,v) \in U,$$

(for some suitable parameter domain U) is given by

$$K = \frac{1}{2ff'}\Big(\frac{1}{1+(f'/g')^2}\Big)'.$$

If the generating curve is parametrized by arc length, show that $K = -f''/f$. Deduce Theorema Egregium in the latter case.

5.2. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface S for which $E = G = 1$ and $F = \cos(uv)$ (so that uv is the angle between the coordinate curves). Determine a suitable parameter domain U on which $\boldsymbol{x}(U)$ is a surface (i.e., where the coordinate curves are not tangential). Show that

$$K = -\frac{1}{\sin(uv)}.$$

5.3. ($\star$) If the coefficients of the first fundamental form of a surface S are given by

$$E = 2 + v^2, \qquad F = 1, \qquad G = 1,$$

show that the Gauss curvature of S is given by

$$K = -\frac{1}{(1+v^2)^2}.$$

5.4. Let $\boldsymbol{x}$ be a local parametrization of a surface S such that $E = 1$, $F = 0$ and G is a function of u only. Show that

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2}$$

and that all the other Christoffel symbols are zero. Hence show that the Gauss curvature K of S is given by

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$