Durham University
Pavel Tumarkin Epiphany 2017

Differential Geometry III, Homework 7 (Week 17)

Due date for starred problems: **Thursday, March 16**.

Curves on surfaces. Geodesics.

7.1. If $\boldsymbol{x}$ is a local parametrization of a surface S in $\mathbb{R}^3$ with $E = 1$, $F = 0$ and G is a function of u only, write down the equations for $s \mapsto \boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ to be a geodesic. Conclude that the coordinate curves, where v is constant, are geodesics.

7.2. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrisation of a surface S, and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a curve parametrised by arclength. Find an expression for the geodesic curvature κ_{g} of $\boldsymbol{\alpha}$ involving u', v', u'', v'', E, F, G, Γ^i_{jk} (i.e. the *geodesic curvature is intrinsic*, κ_{g} depends only on the curve and the first fundamental form of the surface).

7.3. Show that a curve of constant geodesic curvature c on the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a planar circle of length $2\pi(1 + c^2)^{-1/2}$.

Hint: If $\boldsymbol{\alpha}$ is a curve of constant geodesic curvature c show that the vector $\boldsymbol{e}(s) = \boldsymbol{\alpha}(s) \times \boldsymbol{\alpha}'(s) + c\boldsymbol{\alpha}(s)$ does not depend on s, where $(\cdot)'$ denotes differentiation with respect to arc length).

7.4. ($\star$) Let S be a surface in $\mathbb{R}^3$ and suppose that Π is a plane which intersects S orthogonally along a regular curve $\boldsymbol{\gamma}$. If $\boldsymbol{\alpha}(s)$ is a parametrization of $\boldsymbol{\gamma}$ such that $\|\boldsymbol{\alpha}'(t)\|$ is constant, show that $\boldsymbol{\alpha}$ is a geodesic of S.

7.5. (a) Show that any constant speed curve on a surface S in $\mathbb{R}^3$ which is a curve of intersection of S with a plane of reflectional symmetry of S is a geodesic.

(b) Show that the curves of intersection of the coordinate planes in $\mathbb{R}^3$ with the surface S defined by the equation $x^4 + y^6 + z^8 = 1$ are geodesics.

7.6. Let $\boldsymbol{\alpha}$ be a regular curve on a surface S in $\mathbb{R}^3$.

(a) If $\boldsymbol{\alpha}$ is both a line of curvature and a geodesic, show that $\boldsymbol{\alpha}$ is a planar curve.
Hint: Show that $\boldsymbol{N} \times \boldsymbol{\alpha}'$ is constant along $\boldsymbol{\alpha}$).

(b) If $\boldsymbol{\alpha}$ is both a geodesic and a planar curve with nowhere vanishing curvature show that $\boldsymbol{\alpha}$ is a line of curvature.