Durham University
Pavel Tumarkin Epiphany 2017

Differential Geometry III, Homework 8 (Week 18)

Due date for starred problems: **Thursday, March 16**.

Geodesics – 2.

8.1. Find all the geodesics on the flat torus $S^1(1) \times S^1(1) \subset \mathbb{R}^4$, where $S^1(1)$ is the circle of radius 1 in $\mathbb{R}^2$ centered at the origin. Prove that there are infinitely many both closed and non-closed geodesics through the point $(1, 0, 1, 0) \in S^1(1) \times S^1(1)$.

8.2. Let $\mathbb{H}$ be the hyperbolic plane, i.e. the surface $\mathbb{R} \times (0, \infty)$ with coefficients of the first fundamental form $E(u, v) = G(u, v) = 1/v^2$ and $F(u, v) = 0$. Show that the geodesics in $\mathbb{H}$ are the intersections of $\mathbb{H}$ with the lines and circles in $\mathbb{R}^2$ which meet the u-axis orthogonally.

Hint: After obtaining the differential equations you may not try to solve them but, instead, just check that the curves above are indeed geodesics, and then prove that there are no others.

8.3. How many closed geodesics are there on the surface of revolution in $\mathbb{R}^3$ obtained by rotating the curve $z = 1/x^2$, $(x > 0)$ around the z-axis?

8.4. $(\star)$ Let S be the cone obtained by rotating the line $z = \beta x$ $(z > 0)$ around the z-axis, where β is a positive constant. Let $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ be a geodesic on S intersecting the parallel $z = 1$ at an angle ϑ_0. Find the smallest value of $z(s)$. Investigate whether $\boldsymbol{\alpha}$ has self-intersections.

8.5. Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arc length with everywhere non-zero curvature, and let $\boldsymbol{b}(s)$ be a vector such that the map

$$\boldsymbol{x}(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{b}(s), \qquad s \in I, v \in (-\epsilon, \epsilon),$$

is a parametrization of a regular surface S for some $\varepsilon > 0$ (S is a ruled surface — you don't have to show that the surface is regular).

(a) Is the curve $\boldsymbol{\beta} \colon (-\varepsilon, \varepsilon) \longrightarrow S$ given by $\boldsymbol{\beta}(v) = \boldsymbol{x}(s_0, v)$ for some $s_0 \in I$ a geodesic? Justify your answer.

(b) Assume now that $\boldsymbol{b}(s)$ is the binormal of the space curve $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(s)$. Prove that $\boldsymbol{\alpha}$ is a geodesic on S (i.e., show that the *generating* curve is a geodesic on the ruled surface generated by a curve and its binormal.)