Durham University
Pavel Tumarkin

Epiphany 2017

Differential Geometry III, Homework 9 (Week 19)

Gauss–Bonnet Theorem

9.1. Show that the catenoid $x^2 + y^2 = \cosh^2 z$ has a unique closed geodesic.

9.2. Find the Gauss curvature K at all points of the surface given by $x^2 + y^2 = z$. Evaluate $\int_R K \, \mathrm{d}A$, where R is the region of the surface from $z = 0$ to $z = a^2$ ($a \in \mathbb{R}$).

9.3. Verify the Gauss-Bonnet Theorem directly for the region R in the previous question.

9.4. Let S be a connected compact orientable surface in $\mathbb{R}^3$ which is not homeomorphic to a sphere. Prove that S contains elliptic points, hyperbolic points and flat points.

9.5. Let S be a minimal surface homeomorphic to a plane. Show that two geodesics have at most one point of intersection.

Further problem

9.6. Let S be a surface of revolution with local parametrization

$$\mathbf{x}(u, v) = \big(f(v)\cos(u),\ f(v)\sin(u),\ v\big)$$

where $f(v)$ is a positive function of v. Suppose that S is minimal.

(a) Show that f satisfies:

$$\frac{f''(v)}{1 + (f'(v))^2} = \frac{1}{f(v)}.$$

(b) Multiplying both sides of this equation by $2f'(v)$ and integrating, show that for some k:

$$1 + (f'(v))^2 = k^2 f(v)^2$$

(c) Rearranging and integrating, show that for some c

$$f(v) = \frac{1}{k}\cosh(kv + c).$$

The conclusion is that S is (part of) a catenary.