Durham University
Pavel Tumarkin Michaelmas 2016

Differential Geometry III, Homework 2 (Week 2)

Due date for starred problems: **Thursday, October 27**.

Plane curves - 2

2.1. The *catenary* is the plane curve $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{\alpha}(u) = (u, \cosh u)$. It is the curve assumed by a uniform chain hanging under the action of gravity. Sketch the curve. Find its curvature.

2.2. Suppose that $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ is a regular curve, but not necessarily unit speed. Write $\boldsymbol{\alpha}(u) = (x(u), y(u))$. Find the formula for the curvature $\kappa(u)$ at the parameter value u in terms of the functions x and y (and their derivatives) at u.

Hint: consider the corresponding curve $\widetilde{\boldsymbol{\alpha}}$ parametrised by arc length. The curvature $\widetilde{\kappa}$ of $\widetilde{\boldsymbol{\alpha}}$ is then $\widetilde{\kappa}(s) = \widetilde{\boldsymbol{n}}(s) \cdot \widetilde{\boldsymbol{t}}'(s)$, where $\widetilde{\boldsymbol{t}}$ and $\widetilde{\boldsymbol{n}}$ are the unit tangent and unit normal vector of $\widetilde{\boldsymbol{\alpha}}$. Use the relation $\widetilde{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}(\ell^{-1}(s))$, where $s = \ell(u)$ is the arc length, together with the chain rule.

2.3. ($\star$) Compute the curvature of tractrix (see Exercise 1.6) at $\boldsymbol{\alpha}(u)$.

2.4. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a smooth regular plane curve.

(a) Assume that for some $u_0 \in I$ the normal line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ passes through the origin. Show that for some $\epsilon > 0$ the trace $\boldsymbol{\alpha}(u_0 - \epsilon, u_0 + \epsilon)$ can be written in polar coordinates as

$$\boldsymbol{\beta}(\vartheta) = (\rho(\vartheta)\cos\vartheta, \rho(\vartheta)\sin\vartheta)$$

for an appropriate smooth function $\rho(\vartheta)$, where $\vartheta \in J$ for some interval J.

(b) Assume that all normal lines to $\boldsymbol{\alpha}$ pass through the origin. Show that the trace of $\boldsymbol{\alpha}$ is contained in a circle.

(c) Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be given in polar coordinates by

$$\boldsymbol{\alpha}(\vartheta) = (\rho(\vartheta)\cos\vartheta, \rho(\vartheta)\sin\vartheta), \qquad \vartheta \in [a, b]$$

Show that the length of $\boldsymbol{\alpha}$ is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\vartheta$$

(d) In the assumptions of (c), show that the curvature of $\boldsymbol{\alpha}$ is

$$\kappa(\vartheta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[\rho^2 + (\rho')^2]^{3/2}}$$

2.5. Find an arc length parameter for the graphs of the following functions $f, g : (0, \infty) \to \mathbb{R}$:

(a) $f(x) = ax + b, \quad a, b \in \mathbb{R}$;

(b)($\star$) $g(x) = \frac{8}{27}x^{3/2}$.