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Differential Geometry III, Solutions 3 (Week 3)

Evolute and involute

3.1. Let $\boldsymbol{\alpha}$ denote the catenary from Exercise 2.1. Show that

(a) the involute of $\boldsymbol{\alpha}$ starting from $(0,1)$ is the tractrix from Exercise 1.6 (with x- and y-axes exchanged and different parametrization);

(b) the evolute of $\boldsymbol{\alpha}$ is the curve given by

$$\boldsymbol{\beta}(u) = (u - \sinh u \cosh u, 2\cosh u)$$

(c) Find the singular points of $\boldsymbol{\beta}$ and give a sketch of its trace.

Solution:

(a) The involute of $\boldsymbol{\alpha}$ has parametrization

$$\boldsymbol{\gamma}(u) = \boldsymbol{a}(u) - \ell(u)\boldsymbol{t}(u)$$

Since

$$\boldsymbol{a}'(u) = (1, \sinh u),$$

we have

$$\ell(u) = \int_0^u \|\boldsymbol{\alpha}'(v)\|\,\mathrm{d}v = \int_0^u \cosh v\,\mathrm{d}v = \sinh u \qquad \text{and} \qquad \boldsymbol{t}(u) = \frac{1}{\cosh u}(1, \sinh u),$$

so

$$\boldsymbol{\gamma}(u) = \boldsymbol{a}(u) - \sinh u\,\boldsymbol{t}(u) = \left(u - \frac{\sinh u}{\cosh u}, \cosh u - \frac{\sinh^2 u}{\cosh u}\right) = \frac{1}{\cosh u}(u\cosh u - \sinh u, 1)$$

Exchanging coordinate axes, we obtain a curve parametrized by

$$\widetilde{\boldsymbol{\gamma}}(u) = \frac{1}{\cosh u}(1, u\cosh u - \sinh u)$$

The tractrix from Exercise 1.6 is completely characterized by its property (d). Computing the corresponding distance for the curve $\widetilde{\boldsymbol{\gamma}}(u)$ we see that its trace is also a tractrix.

(b) As we have already computed in Exercise 2.1 and in (a),

$$\boldsymbol{t}(u) = \frac{1}{\cosh u}(1, \sinh u), \qquad \kappa(u) = \frac{1}{\cosh^2 u}$$

In particular, $\kappa(u)$ is never zero, and

$$\boldsymbol{n}(u) = \frac{1}{\cosh u}(-\sinh u, 1)$$

Now we can compute the evolute:

$$\boldsymbol{e}(u) = \boldsymbol{\alpha}(u) + \frac{1}{\kappa(u)}\boldsymbol{n}(u) = (u - \sinh u \cosh u, 2\cosh u)$$

as required.

(c) The singular points of $\boldsymbol{e}$ correspond to the vertices of $\boldsymbol{\alpha}$. We have

$$\kappa'(u) = \left(\frac{1}{\cosh^2 u}\right)' = -\frac{2\sinh u}{\cosh^3 u},$$

so $\kappa'(u) = 0$ if and only if $u = 0$. The only singular point of $\boldsymbol{e}$ is $(0, 2)$.

3.2. ($\star$) *Parallels.* Let $\boldsymbol{\alpha}$ be a plane curve parametrized by arc length, and let d be a real number. The curve $\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + d\boldsymbol{n}(u)$ is called the *parallel* to $\boldsymbol{\alpha}$ at distance d.

(a) Show that $\boldsymbol{\beta}$ is a regular curve except for values of u for which $d = 1/\kappa(u)$, where κ is the curvature of $\boldsymbol{\alpha}$.

(b) Show that the set of singular points of all the parallels (i.e., for all $d \in \mathbb{R}$) is the evolute of $\boldsymbol{\alpha}$.

Solution:

(a) Assume $\kappa(u) = 0$ or $d\kappa(u) \neq 1$. The latter is automatically satisfied if $\kappa(u) = 0$. So we just assume that $d\kappa(u) \neq 1$. We need to show that $\boldsymbol{\beta}'(u) \neq 0$. Since $\boldsymbol{\alpha}$ is unit speed, we have

$$\boldsymbol{\beta}'(u) = \boldsymbol{t}(u) + d\boldsymbol{n}'(u) = \boldsymbol{t}(u) + dA\boldsymbol{t}'(u) = \boldsymbol{t}(u) + d\kappa(u)A\boldsymbol{n}(u) =$$
$$= \boldsymbol{t}(u) + d\kappa(u)A^2\boldsymbol{t}(u) = \boldsymbol{t}(u) - d\kappa(u)\boldsymbol{t}(u) = (1 - d\kappa(u))\boldsymbol{t}(u),$$

with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are understood as columns. Note that $\|\boldsymbol{t}(u)\| = 1$, i.e., $\boldsymbol{t}(u) \neq 0$. The initial assumption implies that $(1 - d\kappa(u)) \neq 0$ and, therefore $\boldsymbol{\beta}'(u) \neq 0$, i.e., $\boldsymbol{\beta}(u)$ is regular.

In the case $\kappa(u) \neq 0$ and $d\kappa(u) = 1$, i.e., $d = 1/\kappa(u)$, we obviously have $\boldsymbol{\beta}'(u) = 0$, i.e., $\boldsymbol{\beta}(u)$ is singular.

(b) The evolute is only defined in the case that we have $\kappa(u) \neq 0$ for all u. So we assume this. We have seen in (a) that the singular points of the parallels are precisely those $\boldsymbol{\beta}(u)$ for which we have $d\kappa(u) = 1$, i.e., $d = 1/\kappa(u)$. This means that the set fo singular points of all parallels is

$$\{\boldsymbol{\alpha}(u) + d\boldsymbol{n}(u) \mid u \in I,\ d = 1/\kappa(u)\} = \{\boldsymbol{\alpha}(u) + \frac{1}{\kappa(u)}\boldsymbol{n}(u) \mid u \in I\}$$

which is precisely the parametrization of the evolute of $\boldsymbol{\alpha}$.

3.3. Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth regular curve. Suppose there exists $u_0 \in I$ such that the distance $||\boldsymbol{\alpha}(u)||$ from the origin to the trace of $\boldsymbol{\alpha}$ is maximal at u_0. Show that the curvature $\kappa(u_0)$ of $\boldsymbol{\alpha}$ at u_0 satisfies

$$|\kappa(u_0)| \geq 1/||\boldsymbol{\alpha}(u_0)||$$

Solution:

Note first that the both sides of the inequality we want to prove do not depend on the parametrization, so we may assume without loss of generality that $\boldsymbol{\alpha}$ is parametrized by arc length.

Consider the function $f(u) = \|\boldsymbol{\alpha}\|^2$. Since $f(u)$ has a maximum at u_0, the first derivative of $f(u)$ at u_0 vanishes (cf. Exercise 1.4(b)), and the second derivative is non-positive. Thus, we have

$$0 \geq f''(u_0) = (\boldsymbol{\alpha}(u) \cdot \boldsymbol{\alpha}(u))''|_{u_0} = (2\boldsymbol{\alpha}'(u) \cdot \boldsymbol{\alpha}(u))'|_{u_0} = \boldsymbol{\alpha}''(u_0) \cdot \boldsymbol{\alpha}(u_0) + 2\|\boldsymbol{\alpha}'(u_0)\|^2 = \boldsymbol{\alpha}''(u_0) \cdot \boldsymbol{\alpha}(u_0) + 2$$

To satisfy the inequality above, we must have $\boldsymbol{\alpha}''(u_0) \cdot \boldsymbol{\alpha}(u_0) \leq -1$, which implies $|\boldsymbol{\alpha}''(u_0) \cdot \boldsymbol{\alpha}(u_0)| \geq 1$, and therefore

$$|\kappa(u_0)| = \|\boldsymbol{\alpha}''(u_0)\| \geq 1/\|\boldsymbol{\alpha}(u_0)\|$$

3.4. *Contact with circles.* The points $(x, y) \in \mathbb{R}^2$ of a circle are given as solutions of the equation $C(x, y) = 0$ where

$$C(x, y) = (x - a)^2 + (y - b)^2 - \lambda$$

Let $\boldsymbol{\alpha} = (x(u), y(u))$ be a plane curve. Suppose that the point $\boldsymbol{\alpha}(u_0)$ is also on some circle defined by $C(x, y)$. Then C vanishes at $(x(u_0), y(u_0))$ and the equation $g(u) = 0$ with

$$g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$$

has a solution at u_0. If u_0 is a multiple solution of the equation, with $g^{(i)}(u_0) = 0$ for $i = 1, \ldots, k-1$ but $g^{(k)}(u_0) \neq 0$, we say that the curve $\boldsymbol{\alpha}$ and the circle have *k-point contact* at $\boldsymbol{\alpha}(u_0)$.

(a) Let a circle be tangent to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$. Show that $\boldsymbol{\alpha}$ and the circle have at least 2-point contact at $\boldsymbol{\alpha}(u_0)$.

(b) Suppose that $\kappa(u_0) \neq 0$. Show that $\boldsymbol{\alpha}$ and the circle have at least 3-point contact at $\boldsymbol{\alpha}(u_0)$ if and only if the center of the circle is the center of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(c) Show that $\boldsymbol{\alpha}$ and the circle have at least 4-point contact if and only if the center of the circle is the center of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ and $\boldsymbol{\alpha}(u_0)$ is a vertex of $\boldsymbol{\alpha}$.

Solution:

Denote by $\boldsymbol{c} = (a, b)$ the center of the circle $C(x, y) = 0$. Then the function $g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$ can be written as

$$g(u) = (\boldsymbol{\alpha}(u) - \boldsymbol{c}) \cdot (\boldsymbol{\alpha}(u) - \boldsymbol{c}) - \lambda$$

(a) Differentiating $g(u)$, we obtain

$$g'(u) = 2(\boldsymbol{\alpha}(u) - \boldsymbol{c}) \cdot \boldsymbol{\alpha}'(u)$$

which vanishes if and only if $\boldsymbol{\alpha}'(u)$ is orthogonal to $\boldsymbol{\alpha}(u) - \boldsymbol{c}$. Note that $\boldsymbol{\alpha}(u) - \boldsymbol{c}$ is a radius of the circle, and the vector $\boldsymbol{\alpha}'(u)$ is orthogonal to a radius if and only if it is tangent to the circle.

(b) Differentiating $g'(u)$, we obtain

$$g''(u) = 2(\boldsymbol{\alpha}(u) - \boldsymbol{c}) \cdot \boldsymbol{\alpha}''(u) + 2\|\boldsymbol{\alpha}'(u)\|^2$$

Since $\boldsymbol{\alpha}(u) - \boldsymbol{c}$ is orthogonal to $\boldsymbol{\alpha}'(u)$, it is collinear with $\boldsymbol{\alpha}''(u)$, namely, it is equal to $\pm\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n}$. Assume $\kappa(u) > 0$ (if $\kappa(u) < 0$ the computations are similar), then $\boldsymbol{\alpha}''(u) = -\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n}$. Thus, $g''(u) = 0$ if and only if

$$-2\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n} \cdot \boldsymbol{\alpha}''(u) + 2\|\boldsymbol{\alpha}'(u)\|^2 = 0,$$

which is equivalent to

$$\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\| = \frac{\|\boldsymbol{\alpha}'(u)\|^2}{\boldsymbol{n} \cdot \boldsymbol{\alpha}''(u)}$$

The latter is equal to $1/\kappa(u)$ (see Exercise 2.2).

(c) Again, assume $\kappa(u) > 0$. According to (b), we can write

$$\begin{aligned} g''(u) &= -2\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n} \cdot \boldsymbol{\alpha}''(u) + 2\|\boldsymbol{\alpha}'(u)\|^2 = \\ &= -2\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\kappa(u)\|\boldsymbol{\alpha}'(u)\|^2 + 2\|\boldsymbol{\alpha}'(u)\|^2 = 2\|\boldsymbol{\alpha}'(u)\|^2(1 - \kappa(u)\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|) \end{aligned}$$

Differentiating this expression, we get

$$g'''(u) = 4\boldsymbol{\alpha}''(u) \cdot \boldsymbol{\alpha}'(u)(1 - \kappa(u)\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|) + 2\|\boldsymbol{\alpha}'(u)\|^2(-\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|'\kappa(u) - \|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\kappa'(u))$$

Since the center $\boldsymbol{c}$ of the circle coincides with the center of curvature of $\boldsymbol{\alpha}$, the first summand iz equal to zero. The derivative of $\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|$ is also zero since $\boldsymbol{\alpha}'(u)$ is orthogonal to $\boldsymbol{\alpha}(u) - \boldsymbol{c}$ (cf. (a) or Exercise 1.4(b)). Thus, $g'''(u) = 0$ if and only if $\kappa'(u) = 0$, or, equivalently, $\boldsymbol{\alpha}(u)$ is a vertex of $\boldsymbol{\alpha}$.