Durham University
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Differential Geometry III, Homework 3 (Week 3)

Due date for starred problems: **Thursday, November 10**.

Evolute and involute

3.1. Let $\boldsymbol{\alpha}$ denote the catenary from Exercise 2.1. Show that

(a) the involute of $\boldsymbol{\alpha}$ starting from $(0, 1)$ is the tractrix from Exercise 1.6 (with x- and y-axes exchanged and different parametrization);

(b) the evolute of $\boldsymbol{\alpha}$ is the curve given by

$$\boldsymbol{\beta}(u) = (u - \sinh u \cosh u, 2 \cosh u)$$

(c) Find the singular points of $\boldsymbol{\beta}$ and give a sketch of its trace.

3.2. ($\star$) *Parallels.* Let $\boldsymbol{\alpha}$ be a plane curve parametrized by arc length, and let d be a real number. The curve $\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + d\boldsymbol{n}(u)$ is called the *parallel* to $\boldsymbol{\alpha}$ at distance d.

(a) Show that $\boldsymbol{\beta}$ is a regular curve except for values of u for which $d = 1/\kappa(u)$, where κ is the curvature of $\boldsymbol{\alpha}$.

(b) Show that the set of singular points of all the parallels (i.e., for all $d \in \mathbb{R}$) is the evolute of $\boldsymbol{\alpha}$.

3.3. Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth regular curve. Suppose there exists $u_0 \in I$ such that the distance $||\boldsymbol{\alpha}(u)||$ from the origin to the trace of $\boldsymbol{\alpha}$ is maximal at u_0. Show that the curvature $\kappa(u_0)$ of $\boldsymbol{\alpha}$ at u_0 satisfies

$$|\kappa(u_0)| \geq 1/||\boldsymbol{\alpha}(u_0)||$$

3.4. *Contact with circles.* The points $(x, y) \in \mathbb{R}^2$ of a circle are given as solutions of the equation $C(x, y) = 0$ where

$$C(x, y) = (x - a)^2 + (y - b)^2 - \lambda$$

Let $\boldsymbol{\alpha} = (x(u), y(u))$ be a plane curve. Suppose that the point $\boldsymbol{\alpha}(u_0)$ is also on some circle defined by $C(x, y)$. Then C vanishes at $(x(u_0), y(u_0))$ and the equation $g(u) = 0$ with

$$g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$$

has a solution at u_0. If u_0 is a multiple solution of the equation, with $g^{(i)}(u_0) = 0$ for $i = 1, \ldots, k-1$ but $g^{(k)}(u_0) \neq 0$, we say that the curve $\boldsymbol{\alpha}$ and the circle have *k-point contact* at $\boldsymbol{\alpha}(u_0)$.

(a) Let a circle be tangent to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$. Show that $\boldsymbol{\alpha}$ and the circle have at least 2-point contact at $\boldsymbol{\alpha}(u_0)$.

(b) Suppose that $\kappa(u_0) \neq 0$. Show that $\boldsymbol{\alpha}$ and the circle have at least 3-point contact at $\boldsymbol{\alpha}(u_0)$ if and only if the centre of the circle is the centre of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(c) Show that $\boldsymbol{\alpha}$ and the circle have at least 4-point contact if and only if the centre of the circle is the centre of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ and $\boldsymbol{\alpha}(u_0)$ is a vertex of $\boldsymbol{\alpha}$.