Durham University
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Differential Geometry III, Homework 4 (Week 4)

Due date for starred problems: **Thursday, November 10**.

Space curves - 1

4.1. Check that for two curves $\boldsymbol{\alpha}, \boldsymbol{\beta} : I \to \mathbb{R}^3$ holds

$$(\boldsymbol{\alpha}(u) \times \boldsymbol{\beta}(u))' = \boldsymbol{\alpha}'(u) \times \boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u) \times \boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is the cross-product in $\mathbb{R}^3$.

4.2. ($\star$) Find the curvature and torsion of the curve

$$\boldsymbol{\alpha}(u) = (au, bu^2, cu^3).$$

4.3. ($\star$) Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length.

(a) Determine all regular curves with vanishing curvature κ.

Hint: use Theorem 4.6

(b) Show that if the torsion τ of $\boldsymbol{\alpha}$ vanishes, then the trace of $\boldsymbol{\alpha}$ lies in a plane.

Hint: do NOT use Theorem 4.6

4.4. Assume that $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$, i.e., the trace of $\boldsymbol{\alpha}$ lies in the plane $z = 0$. Calculate the curvature κ of $\boldsymbol{\alpha}$ and its torsion τ. What is the relation of the curvature κ of the space curve $\boldsymbol{\alpha}$ and the (signed) curvature $\overline{\kappa}$ of the plane curve $\overline{\boldsymbol{\alpha}} : I \to \mathbb{R}^2$ defined by $\overline{\boldsymbol{\alpha}}(s) = (x(s), y(s))$ (i.e., the projection of the space curve $\boldsymbol{\alpha}$ to the plane $z = 0$)?

4.5. Consider the regular curve given by

$$\boldsymbol{\alpha}(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

where $a, b, c > 0$ and $c^2 = a^2 + b^2$. The curve $\boldsymbol{\alpha}$ is called a *helix*.

(a) Show that the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) Show that $\boldsymbol{\alpha}$ is parametrized by arc length.

(c) Determine the curvature and torsion of $\boldsymbol{\alpha}$ (and notice that they are both constant).

(d) Determine the equation of the plane through $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$ at each point of $\boldsymbol{\alpha}$ (this plane is called the *osculating plane*).

(e) Show that the line through $\boldsymbol{\alpha}(s)$ in direction $\boldsymbol{n}(s)$ meets the axis of the cylinder orthogonally.

(f) Show that the tangent lines to $\boldsymbol{\alpha}$ make a constant angle with the axis of the cylinder.

Remark: In fact, a helix can be characterized by (a) and (f). If we drop (a), then we obtain a *generalized helix* (see next homework). Another way how to characterize a helix is by (c), i.e., the fact that the curvature and torsion are constant. *Why?*