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Differential Geometry III, Solutions 5 (Week 5)

Space curves - 2

5.1. ($\star$) A curve $\boldsymbol{\alpha}: I \to \mathbb{R}^3$ is called a *(generalized) helix* if its tangent lines make a constant angle with a fixed direction in $\mathbb{R}^3$.

(a) Prove that the curve

$$\boldsymbol{\alpha}(s) = \Big(\frac{a}{c}\int_{s_0}^{s} \sin\vartheta(v)\,\mathrm{d}v, \frac{a}{c}\int_{s_0}^{s} \cos\vartheta(v)\,\mathrm{d}v, \frac{b}{c}s\Big),$$

with $s_0 \in I$, $c^2 = a^2 + b^2$, $a \neq 0$, $b \neq 0$ and $\vartheta'(s) > 0$ is a (generalized) helix.

(b) Assume that $\boldsymbol{\alpha}: I \to \mathbb{R}^3$ is a regular curve with $\tau(s) \neq 0$ for all $s \in I$. Prove that $\boldsymbol{\alpha}$ is a (generalized) helix if and only if κ/τ is constant.

Solution:

(a) We have

$$\boldsymbol{t} = \boldsymbol{\alpha}'(s) = \left(\frac{a}{c}\sin\vartheta(s),\ \frac{a}{c}\cos\vartheta(s),\ \frac{b}{c}\right),$$

so $\|\boldsymbol{t}\| = 1$, that is $\boldsymbol{\alpha}$ is parametrized by arc length.

One way to show that $\boldsymbol{\alpha}$ is a (generalized) helix is to use (b). For this, we compute

$$\boldsymbol{t}' = \boldsymbol{\alpha}'(s) = \Big(\frac{a}{c}\vartheta'(s)\cos\vartheta(s),\ -\frac{a}{c}\vartheta'(s)\sin\vartheta(s), 0\Big) = \frac{a}{c}\vartheta'(s)\left(\cos\vartheta(s), -\sin\vartheta(s), 0\right).$$

We may assume without loss of generality that $\frac{a}{c}\vartheta'(s) > 0$ and take $\kappa(s) = \frac{a}{c}\vartheta'(s)$ and $\boldsymbol{n} = (\cos\vartheta(s), -\sin\vartheta(s), 0)$. Then

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \left(\frac{b}{c}\sin\vartheta(s),\ \frac{b}{c}\cos\vartheta(s), -\frac{a}{c}\right),$$

and

$$\boldsymbol{b}' = \left(\frac{b}{c}\vartheta'(s)\cos\vartheta(s),\ -\frac{b}{c}\sin\vartheta(s), 0\right) = \frac{b}{c}\vartheta'(s)\boldsymbol{n}.$$

Hence $\tau = \frac{b}{c}\vartheta'(s)$ and $\kappa/\tau = \frac{a}{b}$ is constant. It follows from part (b) that $\boldsymbol{\alpha}$ is a generalized helix.

A much simpler way to solve the problem is to guess the vector $\boldsymbol{v}$ such that $\boldsymbol{t}\cdot\boldsymbol{v}$ is constant. Indeed, one can see that z-coordinate of $\boldsymbol{t}$ is equal to b/c, i.e. it is constant. Thus, $\boldsymbol{t}$ makes a constant angle with vector $(0, 0, 1)$, i.e. with z-axis.

(b) We may assume that $\boldsymbol{\alpha}$ is parametrized by arc length. By definition, $\boldsymbol{\alpha}$ is a (generalized) helix if and only if there exists a constant vector $\boldsymbol{v}$ such that

$$\frac{\boldsymbol{t}\cdot\boldsymbol{v}}{\|\boldsymbol{t}\|\,\|\boldsymbol{v}\|} = \frac{\boldsymbol{t}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|} = \text{const}$$

We may assume that $\boldsymbol{v}$ has unit length, so the equality above is equivalent to

$$\boldsymbol{t}\cdot\boldsymbol{v} = \text{const}$$

Equivalently, $\boldsymbol{\alpha}$ is a (generalized) helix if and only if there exists a constant vector $\boldsymbol{v}$ such that

$$\boldsymbol{t}'\cdot\boldsymbol{v} = 0 \iff \boldsymbol{n}\cdot\boldsymbol{v} = 0 \iff \boldsymbol{v} = c\boldsymbol{t} + d\boldsymbol{b}.$$

Since $\boldsymbol{v}$ has unit length, we have $c^2 + d^2 = 1$. Then $\boldsymbol{v}$ makes a constant angle with $\boldsymbol{t}$ if and only if $c = \text{const}$
The vector $\boldsymbol{v}$ is a constant vector if and only if $(c\boldsymbol{t} + d\boldsymbol{b})' = 0$, that is if and only if

$$c'\boldsymbol{t} + c\boldsymbol{t}' + d'\boldsymbol{b} + d\boldsymbol{b}' = c\kappa\boldsymbol{n} + d'\boldsymbol{b} + d\tau\boldsymbol{n} = d'\boldsymbol{b} + (c\kappa + d\tau)\boldsymbol{n} = 0,$$

which holds if and only if

$$d' = c\kappa + d\tau = 0,$$

if and only if $\kappa/\tau = -d/c = \text{const}$

5.2. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be regular curves in $\mathbb{R}^3$ such that, for each u, the principal normals $\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{n}_{\boldsymbol{\beta}}(u)$ are parallel. Prove that the angle between $\boldsymbol{t}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{t}_{\boldsymbol{\beta}}(u)$ is independent of u. Prove also that if the line through $\boldsymbol{\alpha}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\alpha}(u)}$ coincides with the line through $\boldsymbol{\beta}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\beta}(u)}$ then

$$\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + r\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$$

for some real number r.

Solution:
We may assume that one of the curves (say, $\boldsymbol{\alpha}$) is parametrized by arc length. Let

$$f(u) = \boldsymbol{t}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u)$$

We want to show that $f'(u) \equiv 0$.

$$f'(u) = \boldsymbol{t}'_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) + \boldsymbol{t}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}'_{\boldsymbol{\beta}}(u) = \kappa_{\boldsymbol{\alpha}}(u)\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) + \boldsymbol{t}_{\boldsymbol{\alpha}}(u) \cdot \|\boldsymbol{\beta}'(u)\|\kappa_{\boldsymbol{\beta}}(u)\boldsymbol{n}_{\boldsymbol{\beta}}(u) =$$
$$= \boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot (\kappa_{\boldsymbol{\alpha}}(u)\boldsymbol{t}_{\boldsymbol{\beta}}(u) + \lambda(u)\|\boldsymbol{\beta}'(u)\|\kappa_{\boldsymbol{\beta}}\boldsymbol{t}_{\boldsymbol{\alpha}}(u))$$

for the function $\lambda(u)$ defined by $\boldsymbol{n}_{\boldsymbol{\beta}}(u) = \lambda(u)\boldsymbol{n}_{\boldsymbol{\alpha}}$. Now, $\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\alpha}}(u) = 0$, and

$$\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) = \lambda^{-1}(u)\boldsymbol{n}_{\boldsymbol{\beta}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) = 0,$$

so $f'(u) \equiv 0$.
Now assume the lines $\{\boldsymbol{\alpha}(u) + \mu_1\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \mid \mu_1 \in \mathbb{R}\}$ and $\{\boldsymbol{\beta}(u) + \mu_2\boldsymbol{n}_{\boldsymbol{\beta}}(u) \mid \mu_2 \in \mathbb{R}\}$ coincide, i.e.

$$\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u) = \mu(u)\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$$

for some $\mu(u) \in \mathbb{R}$. We want to show that $\mu(u)$ is constant. We can write

$$\mu(u) = \boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot (\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u)),$$

therefore

$$\mu'(u) = \boldsymbol{n}'_{\boldsymbol{\alpha}}(u) \cdot (\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u)) + n_{\boldsymbol{\alpha}}(u) \cdot (\boldsymbol{t}_{\boldsymbol{\alpha}}(u) - \boldsymbol{t}_{\boldsymbol{\beta}}(u))$$

The first summand vanishes since $\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u) = \mu(u)n_{\boldsymbol{\alpha}}(u)$ is parallel to $n_{\boldsymbol{\alpha}}(u)$, and $\boldsymbol{n}'_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{n}_{\boldsymbol{\alpha}}(u) = 0$. The second summand vanishes since $\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$ is parallel to $\boldsymbol{n}_{\boldsymbol{\beta}}(u)$.

5.3. $(\star)$ Let $\boldsymbol{\alpha}$ be the curve in $\mathbb{R}^3$ given by

$$\boldsymbol{\alpha}(u) = e^u(\cos u, \sin u, 1), \qquad u \in \mathbb{R}.$$

If $0 < \lambda_0 < \lambda_1$, find the length of the segment of $\boldsymbol{\alpha}$ which lies between the planes $z = \lambda_0$ and $z = \lambda_1$. Show also that the curvature and torsion of $\boldsymbol{\alpha}$ are both inversely proportional to e^u.

Solution:
We have

$$\boldsymbol{\alpha}'(u) = e^u(\cos u, \sin u, 1) + e^u(-\sin u, \cos u, 0) = (e^u \cos u - e^u \sin u, e^u \sin u + e^u \cos u, e^u),$$

$$\|\boldsymbol{\alpha}'(u)\| = e^u\sqrt{(\cos u - \sin u)^2 + (\sin u + \cos u)^2 + 1} = e^u\sqrt{3}.$$

We first need to find the parameter values when $\boldsymbol{\alpha}$ intersects the planes $z = \lambda_0$ and $z = \lambda_1$. The z-component of $\boldsymbol{\alpha}(u)$ is e^u, so $e^u = \lambda$ implies $u = \ln \lambda$. Then the arc length ℓ between where the curve intersects the planes $z = \lambda_0$ and $z = \lambda_1$ with $0 < \lambda_0 < \lambda_1$ is given by integrating $\|\boldsymbol{\alpha}'(u)\|$ between the corresponding parameter values, namely $u_0 = \ln \lambda_0$ and $u_1 = \ln \lambda_1$. So

$$\ell = \int_{u_0}^{u_1} \|\boldsymbol{\alpha}'(u)\| \,\mathrm{d}u = \int_{u_0}^{u_1} \sqrt{3}e^u \,\mathrm{d}u = \sqrt{3}\big[e^u\big]_{u_0}^{u_1} = \sqrt{3}(e^{u_1} - e^{u_0}) = \sqrt{3}(\lambda_1 - \lambda_0).$$

To compute the curvature we use the formula

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3}.$$

As a result, we obtain

$$\kappa(u) = \frac{\sqrt{2}}{3} \cdot e^{-u}$$

which has the desired form

$$\text{const} \cdot \frac{1}{e^u}.$$

Now one can note that $\boldsymbol{\alpha}$ is a generalized helix: indeed, the cosine of the angle formed by $\boldsymbol{\alpha}'(u)$ with vector $(0,0,1)$ is

$$\frac{(e^u \cos u - e^u \sin u, e^u \sin u + e^u \cos u, e^u) \cdot (0,0,1)}{\sqrt{3}e^u} = \frac{1}{\sqrt{3}}$$

which is constant. Thus, by Exercise 5.1, the torsion is also proportional to $1/e^u$.

Alternatively, one can compute the torsion explicitly to see that

$$\tau(u) = -\frac{1}{3} \cdot e^{-u}$$

which is also of required form.

5.4. Let $\boldsymbol{\alpha}$ be a curve parametrized by arc length with nowhere vanishing curvature κ and torsion τ. Show that if the trace of $\boldsymbol{\alpha}$ lies on a sphere then

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'.$$

Is the converse true?

Solution: Suppose that $\boldsymbol{\alpha}$ lies on the sphere with centre $\boldsymbol{c}$ and radius r. Then

$$(\boldsymbol{\alpha} - \boldsymbol{c}) \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) = r^2 \tag{$*$}$$

Differentiating $(*)$ once we get

$$\boldsymbol{t} \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) = 0.$$

This means that there exist $x, y \in \mathbb{R}$ such that

$$\boldsymbol{\alpha} - \boldsymbol{c} = x\boldsymbol{n} + y\boldsymbol{b}.$$

Differentiating the equality above we obtain

$$\boldsymbol{t} = x'\boldsymbol{n} + x\boldsymbol{n}' + y'\boldsymbol{b} + y\boldsymbol{b}' = x'\boldsymbol{n} + x(-\kappa\boldsymbol{t} - \tau\boldsymbol{b}) + y'\boldsymbol{b} + y\tau\boldsymbol{n} = -x\kappa\boldsymbol{t} + (x' + y\tau)\boldsymbol{n} + (-x\tau + y')\boldsymbol{b}$$

In particular, this implies that

$$-x\tau + y' = 0$$

Let us find x and y. Differentiating $(*)$ twice we get

$$\kappa\boldsymbol{n} \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) + 1 = 0 \tag{$**$}$$

Thus,

$$\kappa x + 1 = 0 \iff x = -\frac{1}{\kappa}.$$

Differentiating $(**)$ we get

$$\kappa' \boldsymbol{n} \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) + \kappa(-\kappa \boldsymbol{t} - \tau \boldsymbol{b}) \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) + \kappa \boldsymbol{n} \cdot \boldsymbol{t} = 0$$

Since $\boldsymbol{n} \cdot \boldsymbol{t} = 0$, this implies

$$\kappa' \boldsymbol{n} \cdot (-\frac{1}{\kappa}\boldsymbol{n} + y\boldsymbol{b}) + \kappa(-\kappa \boldsymbol{t} - \tau \boldsymbol{b}) \cdot (-\frac{1}{\kappa}\boldsymbol{n} + y\boldsymbol{b}) = 0,$$

which gives

$$-\frac{\kappa'}{\kappa} - \kappa\tau y = 0.$$

Hence,

$$y = -\frac{\kappa'}{\kappa^2 \tau}$$

Now the equality $-x\tau + y' = 0$ obtained above becomes

$$\frac{\tau}{\kappa} = \Big(\frac{\kappa'}{\tau\kappa^2}\Big)'.$$

The converse is also true (see e.g. the solution of Exercise 5.5.(b))

5.5. Let $\boldsymbol{\alpha}$ be a regular curve parametrized by arc length with $\kappa > 0$ and $\tau \neq 0$. Denote by $\boldsymbol{n}$ and $\boldsymbol{b}$ the principal normal and the binormal of $\boldsymbol{\alpha}$.

(a) If $\boldsymbol{\alpha}$ lies on a sphere with center $\boldsymbol{c} \in \mathbb{R}^3$ and radius $r > 0$, show that

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\rho \boldsymbol{n} - \rho' \sigma \boldsymbol{b},$$

where $\rho = 1/\kappa$ and $\sigma = -1/\tau$. Deduce that $r^2 = \rho^2 + (\rho'\sigma)^2$.

(b) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that $\boldsymbol{\alpha}$ lies on a sphere of radius r.

Hint: Show that the curve $\boldsymbol{\alpha} + \rho \boldsymbol{n} + \rho' \sigma \boldsymbol{b}$ is constant.

Solution:

(a) Suppose that $\boldsymbol{\alpha}$ lies on the sphere with center $\boldsymbol{c}$ and radius r. From the solution of Exercise 5.4 we know that

$$\boldsymbol{\alpha} - \boldsymbol{c} = x\boldsymbol{n} + y\boldsymbol{b},$$

where

$$x = -\frac{1}{\kappa}, \qquad y = -\frac{\kappa'}{\kappa^2 \tau}$$

We have thus

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\frac{1}{\kappa}\boldsymbol{n} - \frac{\kappa'}{\kappa^2\tau}\boldsymbol{b} = -\rho \boldsymbol{n} - \rho'\sigma \boldsymbol{b},$$

where $\rho = 1/\kappa$ and $\sigma = -1/\tau$. Now,

$$r^2 = (\boldsymbol{\alpha} - \boldsymbol{c}) \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) = (-\rho \boldsymbol{n} - \rho'\sigma \boldsymbol{b}) \cdot (-\rho \boldsymbol{n} - \rho'\sigma \boldsymbol{b}) = \rho^2 + (\rho'\sigma)^2.$$

(b) Suppose that $\rho^2 + (\rho'\sigma)^2 = r^2$. Differentiating we get

$$\rho'(\rho + (\rho'\sigma)'\sigma) = 0.$$

As $\rho' \neq 0$, it follows that

$$\rho + (\rho'\sigma)'\sigma = 0$$

or equivalently,

$$-\rho\tau + (\rho'\sigma)' = 0$$

The curve $\boldsymbol{\alpha} + \rho \boldsymbol{n} + \rho' \sigma \boldsymbol{b}$ is constant (i.e, is a point) if and only if $(\boldsymbol{\alpha} + \rho \boldsymbol{n} + \rho' \sigma \boldsymbol{b})' = 0$. We have,

$$
\begin{aligned}
(\boldsymbol{\alpha} + \rho \boldsymbol{n} + \rho' \sigma \boldsymbol{b})' &= \boldsymbol{t} + \rho' \boldsymbol{n} + \rho \boldsymbol{n}' + (\rho' \sigma)' \boldsymbol{b} + (\rho' \sigma) \boldsymbol{b}' \\
&= \boldsymbol{t} + \rho' \boldsymbol{n} + \rho(-\kappa \boldsymbol{t} - \tau \boldsymbol{b}) + (\rho' \sigma)' \boldsymbol{b} + (\rho' \sigma) \tau \boldsymbol{n} \\
&= (1 - \rho \kappa) \boldsymbol{t} + (\rho' + \rho' \sigma \tau) \boldsymbol{n} + (-\tau \rho + (\rho' \sigma)') \boldsymbol{b} \\
&= 0 \boldsymbol{t} + 0 \boldsymbol{n} + 0 \boldsymbol{b} \\
&= 0.
\end{aligned}
$$

We conclude that $\boldsymbol{\alpha} + \rho \boldsymbol{n} + \rho' \sigma \boldsymbol{b} = \boldsymbol{c}$, for some point $\boldsymbol{c}$. Then

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\rho \boldsymbol{n} - \rho' \sigma \boldsymbol{b}$$

and as $\rho^2 + (\rho' \sigma)^2$ has constant value r^2, $(\boldsymbol{\alpha} - \boldsymbol{c}) \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) = r^2$. This means that the curve $\boldsymbol{\alpha}$ lies on the sphere with center $\boldsymbol{c}$ and radius r.