Durham University
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Differential Geometry III, Solutions 6 (Week 6)

Surfaces - 1

6.1. Let $U \subset \mathbb{R}^2$ be an open set. Show that the set

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0 \text{ and } (x, y) \in U\}$$

is a regular surface.

Solution:

Take $V = \mathbb{R}^3$ and $\boldsymbol{x}(u, v) = (u, v, 0)$. Then all the assumtions of the definition of a surface hold immediately. One could also note that this set is a regular surface as a graph of the zero function.

6.2. Stereographic projection
Let $S^2(1) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ be a 2-dimensional unit sphere. For $(u, v) \in \mathbb{R}^2$, let $\boldsymbol{x}(u, v)$ be the point of intersection of the line in $\mathbb{R}^3$ through $(u, v, 0)$ and $(0, 0, 1)$ with $S^2(1)$ (different from $(0, 0, 1)$).

(a) Find an explicit formula for $\boldsymbol{x}(u, v)$.

(b) Let P be the plane given by $\{z = 1\}$, and for $(x, y, z) \in \mathbb{R}^3 \setminus P$, let $\boldsymbol{F}(x, y, z) \in \mathbb{R}^2$ be such that $(\boldsymbol{F}(x, y, z), 0) \in \mathbb{R}^3$ is the intersection with the (x, y)-plane of the line through $(0, 0, 1)$ and (x, y, z). Show that

$$\boldsymbol{F}(x, y, z) = \frac{1}{1 - z}(x, y).$$

(c) Show that $\boldsymbol{F} \circ \boldsymbol{x} = \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ and deduce that $\boldsymbol{x}$ is a local parametrization of $S^2(1) \setminus \{(0, 0, 1)\}$.

Solution:

(a) We are looking for the intersection of the line L through $(0, 0, 1)$ and $(u, v, 0)$, $(u, v) \in \mathbb{R}^2$, with $S^2(1)$. This line can be parametrized by

$$\boldsymbol{L}(t) := (0, 0, 1) + t\big((u, v, 0) - (0, 0, 1)\big) = (tu, tv, 1 - t), \qquad t \in \mathbb{R}.$$

Now find $t \in \mathbb{R}$ such that $\boldsymbol{L}(t) \in S^2(1)$, i.e., that

$$(tu)^2 + (tv)^2 + (1 - t)^2 = 1.$$

This equation is equivalent to

$$t\big(t(u^2 + v^2 + 1) - 2\big) = 0,$$

hence

$$t = 0 \qquad \text{or} \qquad t = \frac{2}{u^2 + v^2 + 1}.$$

The former solution gives $\boldsymbol{L}(0) = (0, 0, 1)$, and we reject this point. We are looking for the other solution on the sphere, namely

$$\boldsymbol{L}(t) = \Big(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, 1 - \frac{2}{u^2 + v^2 + 1}\Big) = \frac{1}{u^2 + v^2 + 1}\big(2u, 2v, u^2 + v^2 - 1\big).$$

Therefore, $\boldsymbol{x} : U = \mathbb{R}^2 \to S = S^2(1)$ is given by

$$\boldsymbol{x}(u,v) = \frac{1}{u^2+v^2+1}\big(2u, 2v, u^2+v^2-1\big).$$

(b) Let $\boldsymbol{l}$ be the line through $(0,0,1)$ and (x,y,z) where $(x,y,z) \in \mathbb{R}^3 \setminus P$, i.e., $z \neq 1$. Then $\boldsymbol{l}$ can be parametrized by

$$\boldsymbol{l}(t) := (0,0,1) + t\big((x,y,z)-(0,0,1)\big) = (tx, ty, t(z-1)+1), \qquad t \in \mathbb{R}.$$

Its intersection with the xy-plane yields the condition

$$t(z-1)+1 = 0, \quad \text{i.e.,} \quad t = \frac{1}{1-z}$$

on the parameter t (note that $z \neq 1$). Hence the point in the xy-plane is

$$\boldsymbol{l}(1/(1-z)) = \frac{1}{1-z}(x,y,0).$$

In particular, $\boldsymbol{F}$ has the form

$$\boldsymbol{F}(x,y,z) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big)$$

given by the first two coordinates of $\boldsymbol{l}(1/(1-z))$.

(c) We have

$$\begin{aligned}(\boldsymbol{F}\circ\boldsymbol{x})(u,v) &= \boldsymbol{F}(\boldsymbol{x}(u,v)) \\ &= \boldsymbol{F}\Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \underbrace{1-\frac{2}{u^2+v^2+1}}_{z}\Big) \\ &= (u,v)\end{aligned}$$

since $1/(1-z) = (u^2+v^2+1)/2$, hence $\boldsymbol{F}\circ\boldsymbol{x}$ is the identity map on $U = \mathbb{R}^2$.

We conclude now as follows:

(i) $\boldsymbol{x}$ is a smooth map on $U = \mathbb{R}^2$, since its components are rational functions and the denominator never vanishes. Moreover,

$$\boldsymbol{x} : U = \mathbb{R}^2 \to \boldsymbol{x}(U) = S^2(1) \setminus \{(0,0,1)\}, \qquad (*)$$

hence we may choose $V = \mathbb{R}^3 \setminus P$ in the definition of a regular surface (or any other open set containing $S^2(1) \setminus \{(0,0,1)\}$ like $V = \mathbb{R}^2 \times (-\infty, 1)$)

(ii) From $\boldsymbol{F}\circ\boldsymbol{x} = \mathrm{id}_U$ we conclude that $\boldsymbol{x}$ is bijective. The map F is clearly continuous on $\mathbb{R}^3 \setminus \{z=1\}$, therefore $\boldsymbol{x}$ is a homeomorphism.

(iii) The linear independence of $\partial_u\boldsymbol{x}(u,v)$ and $\partial_v\boldsymbol{x}(u,v)$ for all $(u,v) \in U$ is equivalent to

$$\partial_u\boldsymbol{x}(u,v) \times \partial_v\boldsymbol{x}(u,v) \neq 0 \quad \text{for all} \quad (u,v) \in U$$

We have

$$\begin{aligned}\partial_u\boldsymbol{x}(u,v) &= \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2, -2uv, 2u), \\ \partial_v\boldsymbol{x}(u,v) &= \frac{2}{(1+u^2+v^2)^2}(-2uv, 1+u^2-v^2, 2v),\end{aligned}$$

which implies

$$\partial_u\boldsymbol{x}(u,v) \times \partial_v\boldsymbol{x}(u,v) = \frac{4}{(1+u^2+v^2)^3}(-2u, -2v, 1-u^2-v^2) \neq 0$$

Alternatively, we have a quite simple (and abstract) argument for the linear independence of $\partial_u\boldsymbol{x}(u,v)$ and $\partial_v\boldsymbol{x}(u,v)$ for all $(u,v) \in U$,

We have $\boldsymbol{F}\circ\boldsymbol{x} = \mathrm{id}_U$, hence, by the chain rule,

$$\mathrm{d}\boldsymbol{F} \circ \mathrm{d}\boldsymbol{x} = \mathrm{id}_{\mathbb{R}^2}, \quad \text{or pointwise} \quad \big(\mathrm{d}_{\boldsymbol{x}(u,v)}\boldsymbol{F}(\,\mathrm{d}_{(u,v)}\boldsymbol{x}(\boldsymbol{w}))\big) = \boldsymbol{w}$$

for all $\boldsymbol{w} \in \mathbb{R}^2$ and $(u,v) \in U$. This shows that $\mathrm{d}\boldsymbol{x}$ is injective: $\mathrm{d}_{(u,v)}\boldsymbol{x}(\boldsymbol{w}) = 0$ implies by the above equation that $0 = \big(\mathrm{d}_{\boldsymbol{x}(u,v)}\boldsymbol{F}(\,\mathrm{d}_{(u,v)}\boldsymbol{x}(\boldsymbol{w}))\big) = \boldsymbol{w}$.

Now, the image $\mathrm{d}_{(u,v)}\boldsymbol{x}(\mathbb{R}^2)$ of an injective linear map has the same dimension as the preimage, i.e., is two-dimensional. Hence, $\partial_u\boldsymbol{x}(u,v)$ and $\partial_v\boldsymbol{x}(u,v)$ are linearly independent for all $(u,v) \in U$.

6.3. Show that each of the following is a surface:

(a) a cylinder $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$;

(b) a two-sheet hyperboloid given by $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = -1\}$.

In each case find a covering of the surface by coordinate neighborhoods and give a sketch of the surface indicating the coordinate neighbourhoods you have used.

Solution:

(a) Let $U = \{(u,v) \in \mathbb{R}^2 \,|\, |v| < 1\}$. Let us consider four charts:

$$\begin{aligned} V_1 &= \{y > 0\}, & \boldsymbol{x}_1(u,v) &= (v, \sqrt{1-v^2}, u) \\ V_2 &= \{y < 0\}, & \boldsymbol{x}_2(u,v) &= (v, -\sqrt{1-v^2}, u) \\ V_3 &= \{x > 0\}, & \boldsymbol{x}_3(u,v) &= (\sqrt{1-v^2}, v, u) \\ V_4 &= \{x < 0\}, & \boldsymbol{x}_4(u,v) &= (-\sqrt{1-v^2}, v, u) \end{aligned}$$

Clearly, all these maps are smooth, they are homeomorphisms onto their images (check this!), and the images cover the entire cylinder. Linear independence of partial derivatives can be verified by direct computation.

Alternatively, we could deal with two charts only. Namely, let $U = \{(u,v) \in \mathbb{R}^2 \,|\, |v| < \pi\}$, and consider two charts

$$\begin{aligned} V_1 &= \{x > -1\}, & \boldsymbol{x}_1(u,v) &= (\cos v, \sin v, u) \\ V_2 &= \{x < 1\}, & \boldsymbol{x}_2(u,v) &= (-\cos v, -\sin v, u) \end{aligned}$$

One can easily check that all the requirements of the definition of regular surface are satisfied.

(b) Let $U = \mathbb{R}^2$. We parametrize separately the upper half and the lower half of the hyperboloid. For the upper one, we have

$$\boldsymbol{x}(u,v) = (u, v, \sqrt{x^2+y^2+1}$$

For the lower one

$$\boldsymbol{x}(u,v) = (u, v, -\sqrt{x^2+y^2+1}$$

Thus, every half of the hyperboloid is a graph of a smooth function, so it is a regular surface.

6.4. For $a, b > 0$, let

$$S := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \right\}.$$

Show that S is a surface and show that at each point $p \in S$ there are two straight lines passing through p and lying in S (i. e. S is a *doubly ruled* surface).

Solution:

S is a regular surface as a graph of a smooth function $g(x,y) = x^2/a^2 - y^2/b^2$.

Let us find the two lines in S through every point $\boldsymbol{p} \in S$. Let $\boldsymbol{\beta} : \mathbb{R} \to \mathbb{R}^3$ be a line through $\boldsymbol{p} = (x,y,z) \in S$ given by $\boldsymbol{\beta}(s) = \boldsymbol{p} + s\boldsymbol{v}$, where $\boldsymbol{v} = (u,v,w)$. Then

$$\boldsymbol{\beta}(s) = (x + su, y + sv, z + sw)$$

and $\boldsymbol{\beta}(s) \in S$ for every $s \in \mathbb{R}$ if and only if

$$z + sw = \frac{(x+su)^2}{a^2} - \frac{(y+sv)^2}{b^2},$$

which is equivalent to

$$z + sw = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) + \left(\frac{2xu}{a^2} - \frac{2yv}{b^2}\right)s + \left(\frac{u^2}{a^2} - \frac{v^2}{b^2}\right)s^2$$

Since the equality above must hold for every $s \in \mathbb{R}$, the coefficients of the polynomials in the left and right parts of the equality must coincide. Thus, we get

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \qquad w = \frac{2xu}{a^2} - \frac{2yv}{b^2}, \qquad 0 = \frac{u^2}{a^2} - \frac{v^2}{b^2}$$

The first equality holds since $\boldsymbol{p} \in S$. Solving the other two with respect to u, v, w we obtain two vectors (up to scaling)

$$\left(1, \frac{b}{a}, \frac{2x}{a^2} - \frac{2y}{ab}\right) \qquad \text{and} \qquad \left(1, -\frac{b}{a}, \frac{2x}{a^2} - \frac{2y}{ab}\right)$$

giving rise to equations of two lines.

In fact, there is an easier method to find these two lines. We can write the definition of S as

$$z = \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right)$$

Now, for given $\boldsymbol{p} = (x_0, y_0, z_0) \in S$ denote

$$s = \frac{x_0}{a} - \frac{y_0}{b}$$

Then the intersection of the planes

$$\left\{\frac{x}{a} - \frac{y}{b} = s\right\} \qquad \text{and} \qquad \left\{\frac{x}{a} + \frac{y}{b} = \frac{z}{s}\right\}$$

is a line through $\boldsymbol{p}$ lying in S. Similarly, we can denote

$$t = \frac{x_0}{a} + \frac{y_0}{b},$$

and then the intersection of the planes

$$\left\{\frac{x}{a} - \frac{y}{b} = \frac{z}{t}\right\} \qquad \text{and} \qquad \left\{\frac{x}{a} + \frac{y}{b} = t\right\}$$

is also a line through $\boldsymbol{p}$ lying in S. One can easily check that these two lines are distinct.

6.5. ($\star$) Let S be the surface in $\mathbb{R}^3$ defined by $z = x^2 - y^2$. Show that

$$\boldsymbol{x}(u, v) = (u + \cosh v, u + \sinh v, 1 + 2u(\cosh v - \sinh v)), \qquad u, v \in \mathbb{R},$$

is a local parametrization of S. Does $\boldsymbol{x}$ parametrizes the whole surface S?

Solution:

The equality

$$1 + 2u(\cosh v - \sinh v) = (u + \cosh v)^2 - (u + \sinh v)^2$$

can be verified directly, so $\boldsymbol{x}$ is a (clearly, smooth) map from $U = \mathbb{R}^2$ to S. Note that

$$x - y = \cosh v - \sinh v = e^{-v} > 0,$$

so $\boldsymbol{x}$ parametrizes only the part $\{x > y\}$ of the surface.

To show that $\boldsymbol{x}$ is a homeomorphism between $\mathbb{R}^2$ and $S \cap \{x > y\}$ let us compute u and v in terms of (x, y, z). We already know that

$$v = -\ln(x - y)$$

Now we can compute

$$u = x - \cosh v = x - \frac{1}{2}\left(x - y + \frac{1}{x - y}\right) = \frac{1}{2}\left(x + y - \frac{1}{x - y}\right)$$

which is defined (and continuous) for every point of S with $x > y$. Therefore, (u, v) can be defined by $(x, y, z) \in S \cap \{x > y\}$ uniquely, the map is continuous, so $\boldsymbol{x}$ is a homeomorphism.

Linear independence of partial derivatives can be verified by a simple computation. Indeed,

$$\begin{aligned} \partial_u \boldsymbol{x}(u, v) &= (1, 1, \cosh v - \sinh v), \\ \partial_v \boldsymbol{x}(u, v) &= (\sinh v, \cosh v, u(\sinh v - \cosh v)) \end{aligned}$$

which are linearly independent since $\sinh v$ is never equal to $\cosh v$.

6.6. Show that

(a) the cone $\{x^2 + y^2 - z^2 = 0\}$ is not a regular surface;

(b) the one-sheet cone $\{x^2 + y^2 - z^2 = 0,\ z \geq 0\}$ is not a regular surface.

Solution:

(a) Suppose that there exists a homeomorphism $\boldsymbol{x} : U \to S \cap V$ between an open set $U \in \mathbb{R}^2$ and a neighborhood of the origin in the cone (we may assume that both U and V are homeomorphic to a ball). Let $q \in U$ such that $\boldsymbol{x}(q) = (0, 0, 0)$. Then $\boldsymbol{x} : U \setminus \{q\} \to (S \setminus \{(0, 0, 0)\} \cap V$ should also be a homeomorphism. However, $U \setminus \{q\}$ is just a punctured disc, and $S \setminus \{(0, 0, 0)\} \cap V$ is a disjoint union of two punctured discs. Now one can prove that there is no homeomorphism between $U \setminus \{q\}$ and $(S \setminus \{(0, 0, 0)\} \cap V$.

(b) The proof immediately follows from the following (easy) fact: if $S \subset \mathbb{R}^3$ is a regular surface, $\boldsymbol{p} \in S$, then there is a neighborhood V of $\boldsymbol{p}$ in $\mathbb{R}^3$ such that $S \cap V$ is the graph of a smooth function of the form $z = f(x, y)$ or $y = g(x, z)$ or $x = h(y, z)$ (this follows directly from the Implicit Function Theorem, see Proposition 3 in Section 2-2 of Do Carmo's book).