Durham University
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Differential Geometry III, Solutions 7 (Week 7)

Surfaces - 2

7.1. ($\star$) (a) Parametrize the hyperbolic paraboloid S from Exercise 6.4 as a ruled surface (i.e., find a curve $\boldsymbol{\alpha}(v) \subset S$ and a curve $\boldsymbol{w}(v)$ such that $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ will be a parametrization of S).

(b) Now let S be an arbitrary ruled surface, and let $\boldsymbol{x} : J \times I \to \mathbb{R}^3$, $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ be a parametrization of S such that $|\boldsymbol{w}(v)| = 1$ for all $v \in I$, where $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve and I, J are intervals in $\mathbb{R}$. A curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ lying in S is called a *curve of striction* if $\boldsymbol{\beta}'(v) \cdot \boldsymbol{w}'(v) = 0$ for all $v \in I$. Find the curve of striction of the ruled surface in (a) with $a = b = 1$ (using either one of the rulings).

Solution:

(a) Take as $\boldsymbol{\alpha}$ the intersection of the paraboloid with the plane $y = 0$:

$$\boldsymbol{\alpha}(v) = (v, 0, v^2/a^2)$$

From Exercise 6.4 we know that every point $(x, y, z) \in S$ is contained in a line in the direction $(1, b/a, 2x/a^2 - 2y/ab)$, and the line itself is entirely contained in S. Taking $\boldsymbol{\alpha}(v)$ as $(x, y, z) \in S$, we see that the line through $\boldsymbol{\alpha}(v)$ has a direction vector $\boldsymbol{w}(v) = (1, b/a, 2v/a^2)$. Thus, S can be parametrized as

$$\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v) = (v, 0, v^2/a^2) + u(1, b/a, 2v/a^2) = (v + u, ub/a, (v^2 + 2uv)/a^2)$$

(b) If $a = b = 1$, we have a parametrization of the paraboloid

$$\boldsymbol{x}(u,v) = (v, 0, v^2) + u(1, 1, 2v)$$

Normalizing the direction vector computed in (a), we can write this as

$$\boldsymbol{x}(u,v) = (v, 0, v^2) + u\frac{(1, 1, 2v)}{\sqrt{2 + 4v^2}} = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v),$$

so the new (unit) direction vector $\boldsymbol{w}(v) = (1, 1, 2v)/\sqrt{2 + 4v^2}$.

Now we write

$$\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + u(v)\boldsymbol{w}(v),$$

so

$$\boldsymbol{\beta}'(v) = \boldsymbol{\alpha}'(v) + u'(v)\boldsymbol{w}(v) + u(v)\boldsymbol{w}'(v)$$

The assumption $\boldsymbol{\beta}'(v) \cdot \boldsymbol{w}'(v) = 0$ implies

$$0 = \boldsymbol{\beta}'(v)\cdot\boldsymbol{w}'(v) = (\boldsymbol{\alpha}'(v) + u'(v)\boldsymbol{w}(v) + u(v)\boldsymbol{w}'(v))\cdot\boldsymbol{w}'(v) = \boldsymbol{\alpha}'(v)\cdot\boldsymbol{w}'(v) + u'(v)\underbrace{\boldsymbol{w}(v)\cdot\boldsymbol{w}'(v)}_{=0} + u(v)\boldsymbol{w}'(v)\cdot\boldsymbol{w}'(v),$$

so we have

$$u(v) = -\frac{\boldsymbol{\alpha}'(v) \cdot \boldsymbol{w}'(v)}{\|\boldsymbol{w}'(v)\|^2}$$

Let us compute $\boldsymbol{w}'(v)$, and then the numerator and the denominator of the expression above.

$$\boldsymbol{w}'(v) = \left(\frac{(1, 1, 2v)}{\sqrt{2 + 4v^2}}\right)' = \frac{-4v}{(2 + 4v^2)^{3/2}}(1, 1, 2v) + \frac{(0, 0, 2)}{\sqrt{2 + 4v^2}} = -\frac{4}{(2 + 4v^2)^{3/2}}(v, v, -1),$$

so

$$\|\boldsymbol{w}'(v)\|^2 = \frac{8}{(2+4v^2)^2}$$

Since $\boldsymbol{\alpha}'(v) = (1, 0, 2v)$, we have

$$\boldsymbol{\alpha}'(v) \cdot \boldsymbol{w}'(v) = -(1, 0, 2v) \cdot \frac{4}{(2+4v^2)^{3/2}}(v, v, -1) = \frac{4v}{(2+4v^2)^{3/2}},$$

and

$$u(v) = -\frac{\boldsymbol{\alpha}'(v) \cdot \boldsymbol{w}'(v)}{\|\boldsymbol{w}'(v)\|^2} = -\frac{4v}{(2+4v^2)^{3/2}} \Big/ \frac{8}{(2+4v^2)^2} = -\frac{v}{2}(2+4v^2)^{1/2},$$

which implies

$$\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + u(v)\boldsymbol{w}(v) = (v, 0, v^2) - \frac{v}{2}\sqrt{2+4v^2}\frac{(1,1,2v)}{\sqrt{2+4v^2}} = (v, 0, v^2) - \frac{v}{2}(1, 1, 2v) = \frac{v}{2}(1, -1, 0)$$

One can note that $\boldsymbol{\beta}(v)$ is one of the lines from the second family of lines forming S.

7.2. (a) Show that the set S of $(x, y, z) \in \mathbb{R}^3$ fulfilling the equation $xz + y^2 = 1$ is a surface.

(b) Let $\boldsymbol{\alpha}, \boldsymbol{w} : \mathbb{R} \to \mathbb{R}^3$ be given by

$$\boldsymbol{\alpha}(v) = (\cos v, \sin v, \cos v) \quad \text{and} \quad \boldsymbol{w}(v) = (1 + \sin v, -\cos v, -1 + \sin v).$$

Show that for all $v \in \mathbb{R}$ there are two straight lines through $\boldsymbol{\alpha}(v)$, one of which is in direction $\boldsymbol{w}(v)$, both of which lie on S. If $\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$, $u \in \mathbb{R}$, $0 < v < 2\pi$, show that $\boldsymbol{x}$ is a local parametrization of S.

Solution:

(a) Computing the gradient of a smooth function $f(x, y, z) = xz + y^2$ we see that

$$\nabla f(x, y, z) = (z, 2y, x)$$

is equal to zero if and only $(x, y, z) = (0, 0, 0)$, which implies that 1 is a regular value of f, so S is a regular surface.

(b) This can be solved similar to Exercise 6.4. We want to find a line in S through every point $\boldsymbol{\alpha}(v)$, i.e. a vector $\boldsymbol{w}(v) = (a(v), b(v), c(v))$ such that the line $\boldsymbol{\beta}_v(u) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ lies in S. Then

$$\boldsymbol{\beta}_v(u) = (ua + \cos v, ub + \sin v, uc + \cos v)$$

and $\boldsymbol{\beta}_v(u) \in S$ for every $u \in \mathbb{R}$ if and only if

$$(ua + \cos v)(uc + \cos v) + (ub + \sin v)^2 = 1,$$

which is equivalent to

$$u^2(ac + b^2) + u((a + c)\cos v + 2b \sin v) + 1 = 1$$

for every $u \in \mathbb{R}$, which implies

$$a(v)c(v) + b^2(v) = (a(v) + c(v))\cos v + 2b(v)\sin v = 0$$

The equality $(a(v) + c(v))\cos v + 2b(v)\sin v = 0$ implies, up to scaling, that $a(v) + c(v) = 2\sin v$ and $b(v) = -\cos v$. Together with $a(v)c(v) + b^2(v) = 0$ this leads to

$$\boldsymbol{w}(v) = (a(v), b(v), c(v)) = (\pm 1 + \sin v, -\cos v, \mp 1 + \sin v)$$

As in Exercise 6.4, there is an easier way to proceed. Changing coordinates (orthogonally) by $x = (x' - z')/2$ and $z = (x' + z')/2$ we get an equation of a one-sheeted hyperboloid, for which we know that it is doubly ruled.

7.3. Determine all surfaces of revolution which are also ruled surfaces.

Solution:

Let S be such a surface. Since S is a surface of revolution, it contains a circle $\{r(\cos u, \sin u, 0)\}$. Since S is a ruled surface, it contains a line through $\boldsymbol{p} = (r, 0, 0) \in S$ in the direction $\boldsymbol{w} = (a, b, c)$ (we assume that S contains the entire line, otherwise we just get a piece of this surface). Then the whole S is obtained by rotation of the line around z-axis. Therefore, the surface is completely defined by $r > 0$ and a direction (a, b, c). The parameter r does not change the type of S and is responsible for "scaling" only. Let us look how does S depend on (a, b, c).

If the vector (a, b, c) lies in xy-plane (i.e., $c = 0$), then S is not a surface of revolution (since there is no regular curve $\boldsymbol{\alpha}(v)$ in xz-plane). Thus, $c \neq 0$, and we may assume without loss of generality that $\boldsymbol{w} = (a, b, 1)$.

If $a = b = 0$, we get a cylinder

$$x^2 + y^2 = r^2$$

If $a \neq 0, b = 0$, then the line meets z-axis at the point $(0, 0, -r/a)$. Rotating this line around z-axis, we obtain a cone with equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \left(z + \frac{r}{a}\right)^2 = 0$$

(check this!)

If $b \neq 0$, then the line does not meet z-axis, and one can easily see that we get a one-sheeted hyperboloid (shifted along z-axis). Since the hyperbolid is obtained by rotation around z-axis, it should have an equation

$$\frac{x^2}{c^2} + \frac{y^2}{c^2} - (z - d)^2 = k^2$$

for some real numbers c, d and k (check this!). Now, proceeding as in Exercise 7.2(b), we compute an equation to be

$$\frac{x^2}{a^2 + b^2} + \frac{y^2}{a^2 + b^2} - \left(z + \frac{ra}{a^2 + b^2}\right)^2 = \frac{r^2 b^2}{(a^2 + b^2)^2}$$

One can easily check that the line through $(r, 0, 0)$ in the direction $(a, b, 1)$ is contained in S, and thus every rotation of it as well (since the equation is invariant with respect to rotation around z-axis, i.e. with respect to substitution (x, y, z) by $(x \cos u, y \sin u, z)$).

7.4. $(\star)$ Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = (x + y + z - 1)^2$.

(a) Find the points at which grad $f = 0$.

(b) For which values of c the level set $S := \{p = (x, y, z) \in \mathbb{R}^3 \mid f(p) = c\}$ is a surface?

(c) What is the level set $f(p) = c$?

(d) Repeat (a) and (b) using the function $f(x, y, z) = xyz^2$.

Solution:

(a) $\nabla f(x, y, z) = 2(x + y + z - 1)(1, 1, 1)$, which implies that $\nabla f = 0$ if and only if $x + y + z = 1$.

(b) According to (a), $\nabla f = 0$ if and only if $x + y + z - 1 = 0$, which is equivalent to $f(x, y, z) = 0$. Thus, the only singular value of f is 0, and for any $c \neq 0$ the level set $f(p) = c$ is a regular surface.

However, although $c = 0$ is a singular value of f, for $c = 0$ the level set $f(p) = c$ is also a regular surface: $f(p) = 0$ is a plane $x + y + z = 1$ which is clearly regular.

(c) The equation $(x + y + z - 1)^2 = c$ is equivalent to $(x + y + z - 1) = \pm\sqrt{c}$, so it is a union of two parallel planes for $c \neq 0$, and one plane for $c = 0$.

(d) $\nabla f(x, y, z) = (yz^2, xz^2, 2xyz) = z(yz, xz, 2xy)$, which implies that $\nabla f = 0$ if and only if $z = 0$ or $x = y = 0$, so the only singular value of f is 0, and for any $c \neq 0$ the level set $f(p) = c$ is a regular surface. The level set $f(p) = 0$ is a union of three coordinate planes, so it is not a regular surface (the "bad" points are ones lying on coordinate axes, check this!)

7.5. Möbius band

Let S be the image of the function $f : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3, (\varepsilon > 0)$, defined by

$$f(u,v) = \left(\left(2 - v \sin\frac{u}{2}\right) \sin u,\ \left(2 - v \sin\frac{u}{2}\right) \cos u,\ v \cos\frac{u}{2} \right).$$

Show that, for ε sufficiently small, S is a surface in $\mathbb{R}^3$ which may be covered by two coordinate neighborhoods. Give a sketch of the surface indicating the curves $u =$ const and $v =$ const (such curves are called *coordinate curves*).

Solution:

(a) Let us write $f(u,v)$ as

$$f(u,v) = \underbrace{(2\sin u, 2\cos u, 0)}_{=:\boldsymbol{\alpha}(u)} + v \underbrace{\left(-\sin\frac{u}{2}\sin u,\ -\sin\frac{u}{2}\cos u,\ \cos\frac{u}{2}\right)}_{=:\boldsymbol{w}(u)}$$

By the form of f, S is a ruled surface, and one can easily see that $\boldsymbol{\alpha}(u)$ is regular, and $\boldsymbol{\alpha}'(u)$ and $\boldsymbol{w}(u)$ are not collinear for all $u \in \mathbb{R}$. Now, to have a regular surface, we need the intervals through different points of $\boldsymbol{\alpha}(u)$ to be disjoint. A straightforward calculation shows that this holds for small ε (say, for $0 < \varepsilon < 2$).

In fact, the latter can be shown geometrically. One can note that the line $\boldsymbol{l}_u(v) = f(u,v)$ through $\boldsymbol{\alpha}(u)$ in the direction $\boldsymbol{w}(u)$ meets the z-axis at the point $(0, 0, \cot u/2)$ (unless $u = 0$: in this case $\boldsymbol{l}_u(v)$ is parallel to z-axis). Therefore, if two such lines $\boldsymbol{l}_{u_1}(v)$ and $\boldsymbol{l}_{u_2}(v)$ intersect, they should be contained in a plane passing through the z-axis, and thus intersect the circle $\{x^2 + y^2 = 4,\ z = 0\}$ (which is the trace of $\boldsymbol{\alpha}$) in two opposite points only, which is clearly not the case (unless $u_2 = u_1 + n\pi$) since the lines meet $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_1)$ and $\boldsymbol{\alpha}(u_2)$. The condition that the intervals lying on lines $\boldsymbol{l}_u(v)$ and $\boldsymbol{l}_{u+\pi}(v)$ do not intersect is guaranteed by the assumption $\varepsilon < 2$.

(b) Clearly, f is not injective: $\boldsymbol{\alpha}$ has a period 2π, so $f(u_0 + 2\pi, 0) = f(u_0, 0)$. However, if we take an open set $U_1 = (0, 2\pi) \times (-\varepsilon, \varepsilon)$, then the restriction of f on U_1 is injective, and the image of U_1 is the whole Möbius strip except one interval $f(0 \times (-\varepsilon, \varepsilon))$. Taking $U_2 = (-\pi, \pi) \times (-\varepsilon, \varepsilon)$, we see that $f(U_1) \cup f(U_2) = S$.